AN ELEMENTARY PROOF OF A THEOREM CONCERNING INFINITELY CONNECTED DOMAINS

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Abstract. Using classical complex function theory, it is shown that any infinitely connected plane domain is conformally equivalent to a domain whose isolated boundary components are analytic Jordan curves. This allows an elementary proof to be given of the result that a domain with countably many boundary components is conformally equivalent to a domain bounded by analytic Jordan curves.

1. It is an immediate consequence of the Riemann mapping theorem that any domain of finite connectivity can be mapped conformally onto a domain whose boundary components are analytic Jordan curves. In recent work ([2], [3]) the author has shown that this result holds for infinitely connected domains (with countably many boundary components). The proof, by transfinite induction, uses only the standard theory of normal families and the following: every plane domain is conformally equivalent to a domain whose isolated boundary components are analytic Jordan curves. (Points are to be considered as degenerate analytic Jordan curves, but in any case if they are isolated they can be ignored.) The proofs given in [2] and [3] of this statement use deep results in the theory of quasiconformal mappings. We give here a direct "normal family proof" (a simplified version of the transfinite induction argument in [3]). Once this is done, the entire proof of the general theorem (see §3) becomes elementary—using only classical function theory (cf. [3, Note, p. 418]).

2. We recall the classical proof for finitely connected domains. If the boundary components of such a domain \( D_0 \) are indexed by \( k = 1, \ldots, n \) and this indexing is preserved under conformal maps of the domain, then one obtains maps \( f_k (0 \leq k \leq n) \) of \( D_0 \) onto domains \( D_k \) whose first \( k \) boundary components are analytic Jordan curves. This is done by successively applying the Riemann mapping theorem to the domain bounded

\( D_{k-1} \) and the \( k \)th component of the boundary of \( D_{k-1} \).
by the $k$th boundary component of $D_{k-1}$. This can always be done leaving
invariant two fixed points. If $h^*_{k-1}$ is this map and $h_{k-1}$ the restriction of
$h^*_{k-1}$ to $D_{k-1}$ we let $f_k = h_{k-1} \circ f_{k-1}$ and $D_k = f_k(D_0)$. The first $k-1$ boundary
components of $D_k$ are analytic Jordan curves because $h_{k-1}$ is the restriction
of a map which is conformal in a neighborhood of these components. The
$k$th boundary component of $D_k$ is the unit circle.

These remarks can be applied to an infinitely connected domain $D_0$
whose (nondegenerate) isolated boundary components (of which there are
at most countably many) are indexed by the natural numbers. With the
above notations let, for $1 < j < \infty$, $h^*_i = h^*_{i-1} \circ \cdots \circ h^*_{1}$ so that $h^*_k$ is con-
formal in the domain bounded by those boundary components of $D_i$
indexed by $i+1, \cdots, j$ and let $h_{ij}: D_i \to D_j$ be the restriction of $h^*_i$ to $D_i$.
We obtain then the

**Lemma.** Let $D_0$ be an arbitrary domain. Then there exists a collection of
domains $D_i$ ($i = 1, 2, \cdots$), maps $f_i: D_0 \to D_i$ and (for $i < j$) maps $h_{ij}: D_i \to D_j$
satisfying:

1. The first $i$ boundary components of $D_i$ are analytic Jordan curves.
2. $h_{ij}$ is the restriction to $D_i$ of a map $h^*_i$ of the domain bounded by the
   boundary components of $D_i$ indexed by $i+1, \cdots, j$.
3. $h_{ij} = h_{ik} \circ h_{jk}$ ($i < k < j$).
4. $f_i = h_{ij} \circ f_j$.

(A formal induction proof could easily be given.)

We state the

**Proposition [3].** An arbitrary plane domain is conformally equivalent
to a domain whose isolated boundary components are analytic Jordan curves.

**Proof.** The maps $h^*_i$ are univalent in the domain $D^*_i$ bounded by all
the boundary components of $D_i$ except those indexed by $1, \cdots, i$.
Consequently, they form a normal family (the distortion theorem implies
that they are locally uniformly bounded—see e.g. [1]). A diagonalization
argument shows that there is a subsequence $f_n$ such that $h^*_{i_n}$ converges
(uniformly on compact subsets of $D^*_i$) for each $i$ to a map $h^*_{i\infty}$ which is
univalent in $D^*_i$. (It is not a constant because of the normalization on
$h^*_{i\infty}$.) If $h_{i\infty}$ is the restriction of $h^*_{i\infty}$ to $D_i$ then, for $i < k < \infty$, $h_{i\infty} = h_{k\infty} \circ h_{ik}$
and, by (4), $f_{i_n} \to f_{i\infty}: D_0 \to D_{i\infty}$ where $f_{i\infty} = h_{i\infty} \circ f_i$ for every $i$. 

\[
\begin{array}{c}
D_0 \\
\downarrow f_{i\infty}
\end{array}
\begin{array}{c}
D_i \\
\leftarrow h_{i\infty}
\end{array}
\begin{array}{c}
D_{i\infty} \\
\rightarrow f_{i\infty}
\end{array}
\]
For \( i > l \) the \( l \)th boundary component of \( D_1 = f_i(D_\infty) \) is an analytic Jordan curve and \( h_{i, \infty} \) is the restriction of a map conformal in a neighborhood of the first \( i \) boundary components of \( D_1 \) (in particular, in a neighborhood of the \( l \)th). Hence the \( l \)th boundary component of \( D_\infty \) is an analytic Jordan curve for arbitrary \( l \) and the proposition is proved.

Note. The stronger result that any domain is conformally equivalent to a domain whose isolated boundary components are circles was obtained in [2] with the use of quasiconformal mappings. It also can be obtained as above, using the result for finitely connected domains and the reflection principle.

3. For completeness, we recall briefly how the above proposition is used in the proof of the general theorem: a domain with countably many boundary components is conformally equivalent to a domain bounded by analytic Jordan curves [3].

The collection \( \Gamma_1 \) of boundary components is provided with a natural metric space structure. One considers the chain \( \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_\alpha \supseteq \cdots \) indexed by the ordinal numbers where if the ordinal \( \alpha \) has a predecessor \( \alpha - 1 \) then \( \Gamma_\alpha \) is the derived set of \( \Gamma_{\alpha - 1} \) and if \( \alpha \) is a limit ordinal then \( \Gamma_\alpha = \bigcap \Gamma_\beta, \beta < \alpha \).

Next, one considers an induction hypothesis analogous to (1)–(4) above. Here again, it is the richness of the hypotheses which is the key to the method of proof.

The step to an ordinal \( \alpha \) with a predecessor is made by applying the above proposition to the domain bounded by \( \Gamma_{\alpha - 1} \), observing that the isolated boundary components of this domain are \( \Gamma_{\alpha - 1} - \Gamma_\alpha \).

For a limit ordinal, one uses a normal family argument, similar to that used in the proof of the proposition of §2, but now on a sequence of ordinals tending to the limit ordinal.

REFERENCES


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