ON POLYNOMIAL APPROXIMATION IN $A_q(D)$

THOMAS A. METZGER

Abstract. Let $D$ be a bounded Jordan domain with rectifiable boundary and define $A_q(D)$, the Bers space, as the space of holomorphic functions $f$, such that

$$\int_D |f|^q \lambda_D^{2-q} \, dx \, dy$$

is finite, where $\lambda_D$ is the Poincaré metric for $D$. It is shown that the polynomials are dense in $A_q(D)$ for $q > 3/2$.

1. Introduction. Let $D$ be a bounded Jordan domain. Define $A_q(D)$ ($q > 1$), the Bers space, as the Banach space of holomorphic functions $f(z)$, such that

$$\|f\|_q = \int_D |f(z)|^q \lambda_D^{2-q}(z) \, dx \, dy < \infty,$$

where $\lambda_D(z)$ is the Poincaré metric for $D$. If $\phi$ is a Riemann mapping function from $D$ onto $U$, the unit disk, and $\psi = \phi^{-1}$, then

$$\int_D \lambda_D^{2-q}(z) \, dx \, dy = \int_U |\psi'(\zeta)|^q (1 - |\zeta|^2)^{q-2} \, d\zeta \, dn.$$

Since for Jordan domains the rectifiability of the boundary is equivalent to $\psi \in H^1(U)$, the Hardy class (cf. [3, p. 44]), it follows immediately by a theorem of Carleson [3, p. 157] that if the boundary of $D$ is rectifiable then (1.2) is finite for all $q > 1$. Hence $D$ bounded implies that the polynomials belong to $A_q(D)$, for all $q > 1$.

The question of polynomial density in $A_q(D)$ has been considered by various authors. In the case $q \geq 2$, Bers [2] and Knopp [4] proved that the polynomials were dense in $A_q(D)$ without any assumptions on the mapping functions $\psi$ or $\phi$. Later Sheingorn [6] proved that the polynomials are dense in $A_q(U^*)$, for all $q > 1$, where $U^*$ is a particular Jordan domain which appears in the proof of the "Main Lemma" of Knopp [4]. Finally Metzger...
and Sheingorn [5] proved that if either \( \psi' \in H^p \), for some \( p > 1 \), or \( D \) is a Smirnov domain, then the polynomials are dense in \( A_q(D) \), for all \( q > 1 \).

In this paper we show

**Theorem 1.** If \( D \) is a bounded Jordan domain with a rectifiable boundary, then the polynomials are dense in \( A_q(D) \), for \( q > 3/2 \).

2. **Proof of Theorem 1.** Clearly we can assume that the origin lies in \( D \) and we choose \( \psi(z) \), such that \( \psi(0) = 0, \psi'(0) > 0 \). We now note

**Lemma 1.** Suppose \( D \) is a bounded Jordan domain with rectifiable boundary. The polynomials are dense in \( A_q(D) \) if and only if \( \psi^q \) can be approximated by polynomials in \( A_q(D) \).

The necessity is clear since \( \psi^q \in A_q(D) \) for all \( q > 1 \) and the sufficiency follows easily from Lemma 1 of [5].

We define \( \mathcal{H}_p(D) = \{ f : f \) is holomorphic on \( D \) and \( \| f \|^*_p < \infty \} \), where

\[
\| f \|^*_p = \left( \int_D |f|^p \, dx \, dy \right)^{1/p}.
\]

An application of H"older's inequality yields

\[
\| F \|_q = \int_U |F \circ \psi| |\psi^q| (1 - |z|^2)^{p-2} \, dx \, dy
\]

\[
\leq \left( \int_U |F \circ \psi|^p |\psi^q|^q \, dx \, dy \right)^{1/p}
\cdot \left( \int_U |\psi'|^{(qp-2)/(p-1)} (1 - |z|^2)^{(qp-2p)/(p-1)} \, dx \, dy \right)^{(p-1)/p}
\]

\[
= \| F \|^*_p \left( \int_D \lambda_D^{(qp-2)/(p-1)} \, dx \, dy \right)^{(p-1)/p} < \infty,
\]

if \( (qp-2)/(p-1) > 1 \), i.e. if \( p > 1/(q-1) \), by (1.2) and the fact that \( (qp-2)/(p-1)-2 = (qp-2p)/(p-1) \). Thus polynomial approximation in \( \mathcal{H}_p(D) \) implies polynomial approximation in \( A_q(D) \) and, by Lemma 1, it follows that we need only show that \( \psi' \in \mathcal{H}_p(D) \) for \( p > q/(q-1) \).

To see that this holds, we first note that \( q < 2 \) implies \( p > 2 \) and by changing variables we get

\[
\int_D |\psi'|^p \, dx \, dy = \int_U |\psi'|^2 \, dx \, dy = I.
\]
Since \( \psi \) is a bounded schlicht function with \( \psi(0) = 0 \), it follows that there exists an \( M > 0 \) such that \( |\psi'(z)| \geq M(1 - |z|^2) \) for all \( z \) in \( U \). Hence \( I \) is finite if \( p < 3 \), i.e., if \( q > 3/2 \) and this completes the proof.

**Remark.** If one lets \( G \) be a discrete group of conformal transformations on \( D \) and defines \( A_q(D, G) \) as in Bers [1] then Theorem 2 of that paper yields the density of the Poincaré theta series of the polynomials in \( A_q(D, G) \). (Cf. [1] and [5] for a precise formulation of the result plus all definitions and notations.)

**Bibliography**


**Department of Mathematics, Texas A&M University, College Station, Texas 77843**