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MAXIMAL SUBLATTICES OF FINITE DISTRIBUTIVE LATTICES

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ABSTRACT. A best possible estimate is established for the size and length of maximal proper sublattices of finite distributive lattices.

1. **Introduction.** Papers of H. Sharp [3] and D. Steven [4] have established the following result:

THEOREM 1. *If L is a finite Boolean lattice with $|L|\geqq 4$ and M is a maximal proper sublattice of L then* (i) $|M|=\frac{3}{4}|L|$ *and* (ii) $l(M)=l(L)$, *where $l(L)$ denotes the length of L.*

The purpose of this note is to prove an analogous result for finite distributive lattices, as well as to provide a simple alternative proof of Theorem 1.

THEOREM 2. *If L is a finite distributive lattice with $|L|\geqq 3$ and M is a maximal proper sublattice of L then* (i) $|M|\geqq\frac{2}{3}|L|$ *and* (ii) $l(M)\geqq l(L)-1$.

Furthermore, these inequalities are best possible in the sense that for every integer $n\geqq 1$ there is a distributive lattice L_n with a maximal proper sublattice M_n such that $|L_n|=3n$, $|M_n|=2n$ and $l(M_n)=l(L_n)-1$ (see Figure 1).

2. **Preliminaries.** Let $J(L)=\{x\in L|x$ join-irreducible$\}$, $M(L)=\{x\in L|x$ meet-irreducible$\}$ and for all further notation and terminology refer to [2]. Recall that in lattices of finite length every element can be expressed as the join of all join-irreducibles contained in it and dually.

The basic result we need is that characterizing maximal proper sublattices of an arbitrary finite distributive lattice [1]. In what follows we provide a new simple proof of this result.

LEMMA 1. *If L is a lattice of finite length and S is a proper sublattice then there exist $a\in J(L)$, $b\in M(L)$, $a\leqq b$, such that $S\cap[a, b]=\varnothing$.*

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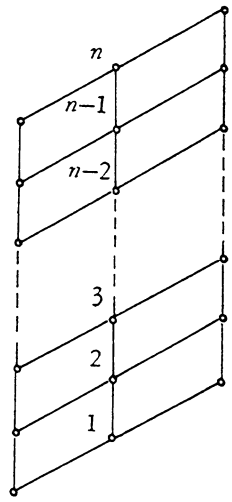

L_n, $n \geq 1$

$M_n = L_n - \{1, 2, \cdots, n\}$

FIGURE 1

PROOF. Since S is a proper sublattice there exists $a \in J(L) - S$. Let $B = \{b \in M(L) | a \leqq b\}$; clearly $B \neq \varnothing$. If for every $b \in B$ there were some $x_b \in S \cap [a, b]$ then $\bigwedge (x_b | b \in B) \in S$ since S is a complete sublattice. On the other hand, $a \leqq \bigwedge (x_b | b \in B) \leqq \bigwedge B = a$, that is, $\bigwedge (x_b | b \in B) = a \notin S$, which is a contradiction. Thus, we can conclude that there is some $b \in B$ such that $S \cap [a, b] = \varnothing$.

LEMMA 2. *If L is a distributive lattice, $a \in J(L)$, $b \in M(L)$, and $a \leqq b$, then $L - [a, b]$ is a sublattice of L.*

PROOF. Let us assume that there exist x, $y \in L - [a, b]$ such that $x \vee y \in [a, b]$. Since $a \in J(L)$ and L is distributive we get that $a \leqq x$ or $a \leqq y$, so that either $x \in [a, b]$ or $y \in [a, b]$, contradicting our choice.

We can now give the basic characterization [**1**].

THEOREM 3. *If L is a finite distributive lattice and M is a maximal proper sublattice, then there exist $a \in J(L)$, $b \in M(L)$, $a \leqq b$, such that* (i) $L - M = [a, b]$, (ii) $(a, b] \subseteq L - J(L)$ *and* (iii) $[a, b) \subseteq L - M(L)$.

PROOF. (i) By Lemma 1 there exist $a \in J(L)$, $b \in M(L)$, $a \leqq b$, such that $M \cap [a, b] = \varnothing$, that is, $M \subseteq L - [a, b] \subset L$. By Lemma 2, $L - [a, b]$ is a sublattice so that by the maximality of M, $M = L - [a, b]$.

(ii) If there were some $c \in (a, b]$ such that $c \in J(L)$, then in view of Lemma 2, $M = L - [a, b] \subseteq L - [c, b] \subset L$ is a chain of sublattices contradicting the maximality of M. (iii) follows dually.

We are now ready to prove Theorems 1 and 2.

3. **The Boolean case.** Suppose M is a maximal proper sublattice of the Boolean lattice $\mathbf{2}^n$. Without loss of generality we may take $n \geqq 3$ in which case it must be that $0, 1 \in M$. By Theorem 3 there exist $a \in J(L)$, $b \in M(L)$ such that $L - M = [a, b]$. But then a must be an atom and b a coatom, so that $[a, b] \cong \mathbf{2}^{n-2}$, and this, of course, implies that $|M| = \frac{3}{4}|\mathbf{2}^n|$. Finally, if a' is the complement of a, then $[0, a'] \cup [a', 1] \subseteq M$, since $0, 1 \in M$ and $L - M$ is a convex sublattice of L. Since all maximal chains have the same order, $l(M) = l(L)$ which completes the proof of Theorem 1.

4. **The distributive case.** The proof of Theorem 2 amounts essentially to an enumeration of sufficiently many joins and meets of subsets of $J(L)$ and $M(L)$, respectively.

We can certainly assume that $|L - M| \geqq 2$ so that by the maximality of M, $0, 1 \in M$ and $L - M \subseteq L - (J(L) \cap M(L))$. By Theorem 3 there exist $a \in J(L)$, $b \in M(L)$, $a \leqq b$, such that $L - M = [a, b]$.

Now, L is finite, so in particular, for every $x \in (a, b]$ there exists $A \subseteq J(L)$ such that $x = \bigvee A$. If $A \subseteq M$ then $\bigvee A \in M$ since M is a finite sublattice. Therefore, $A \cap [a, b] \neq \varnothing$, and, in fact, by Theorem 3, $J(L) \cap [a, b] = \{a\}$, so that $a \in A$. This shows that for every $x \in (a, b]$ we can select a subset $f(x)$ in $J(L) \cap M$ such that $x = a \vee \bigvee f(x)$, where $\bigvee f(x) \in M$; in fact, $\bigvee f(x) \in M - \{0, 1\}$. Moreover, it is immediate that if $x, y \in (a, b]$, $\bigvee f(x) = \bigvee f(y)$ if and only if $x = y$. Thus, we can conclude that

$$\{\bigvee f(x) \mid x \in (a, b]\} \subseteq M - \{0, 1\}$$

and

$$|\{\bigvee f(x) \mid x \in (a, b]\}| = |(a, b]| = |L - M| - 1.$$

Dually, we get a choice function g from $[a, b)$ into the set of subsets of $M(L)$ such that $y = b \wedge \bigwedge g(y)$, for every $y \in [a, b)$, $\{\bigwedge g(y) | y \in [a, b)\} \subseteq M - \{0, 1\}$, and $|\{\bigwedge g(y) | y \in [a, b)\}| = |L - M| - 1$.

We now claim that $\{\bigvee f(x) | x \in (a, b]\} \cap \{\bigwedge g(y) | y \in [a, b)\} = \varnothing$. In fact, suppose there were suitable $x, y \in [a, b]$ such that $\bigvee f(x) = \bigwedge g(y)$. Clearly, $b \geqq x = a \vee \bigvee f(x) \geqq \bigvee f(x)$ so that $\bigvee f(x) = b \wedge \bigvee f(x) = b \wedge \bigwedge g(y) = y$, which is impossible since $\bigvee f(x) \in M$ while $y \in L - M$.

It now follows that $|L| \geqq |L - M| + |\{\bigvee f(x) | x \in (a, b]\}| + |\{\bigwedge g(y) | y \in [a, b)\}| + |\{0, 1\}| = 3|L - M|$, from which (i) is an immediate consequence.

To establish (ii) take some $c \in L$ such that $b > c \geqq \bigvee f(b)$ (b covers c). If $c \in [a, b]$ then $c \geqq \bigvee f(b)$ and $c \geqq a$ together imply that $b > c \geqq a \wedge \bigvee f(b) = b$, which is impossible. Thus, $c \in M$ and by the convexity of $L - M$,

$[0, c] \subseteq M$. Finally, taking a maximal chain in $[0, c]$, adjoining to it a maximal chain in $(b, 1]$ and recalling that in distributive lattices all maximal chains have the same order, we can conclude that $l(M) \geqq l(L) - 1$, completing the proof of Theorem 2.

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