MAXIMAL SUBLATTICES OF FINITE DISTRIBUTIVE LATTICES

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Abstract. A best possible estimate is established for the size and length of maximal proper sublattices of finite distributive lattices.

1. Introduction. Papers of H. Sharp [3] and D. Steven [4] have established the following result:

Theorem 1. If \( L \) is a finite Boolean lattice with \( |L| \geq 4 \) and \( M \) is a maximal proper sublattice of \( L \) then (i) \( |M| = \frac{3}{2}|L| \) and (ii) \( l(M) = l(L) \), where \( l(L) \) denotes the length of \( L \).

The purpose of this note is to prove an analogous result for finite distributive lattices, as well as to provide a simple alternative proof of Theorem 1.

Theorem 2. If \( L \) is a finite distributive lattice with \( |L| \geq 3 \) and \( M \) is a maximal proper sublattice of \( L \) then (i) \( |M| \geq \frac{3}{2}|L| \) and (ii) \( l(M) \geq l(L) - 1 \).

Furthermore, these inequalities are best possible in the sense that for every integer \( n \geq 1 \) there is a distributive lattice \( L_n \) with a maximal proper sublattice \( M_n \) such that \( |L_n| = 3n \), \( |M_n| = 2n \) and \( l(M_n) = l(L_n) - 1 \) (see Figure 1).

2. Preliminaries. Let \( J(L) = \{ x \in L | x \text{ join-irreducible} \} \), \( M(L) = \{ x \in L | x \text{ meet-irreducible} \} \) and for all further notation and terminology refer to [2].

Recall that in lattices of finite length every element can be expressed as the join of all join-irreducibles contained in it and dually.

The basic result we need is that characterizing maximal proper sublattices of an arbitrary finite distributive lattice [1]. In what follows we provide a new simple proof of this result.

Lemma 1. If \( L \) is a lattice of finite length and \( S \) is a proper sublattice then there exist \( a \in J(L) \), \( b \in M(L) \), \( a \leq b \), such that \( S \cap [a, b] = \emptyset \).

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PROOF. Since $S$ is a proper sublattice there exists $a \in J(L) - S$. Let $B = \{b \in M(L) | a \leq b\}$; clearly $B \neq \emptyset$. If for every $b \in B$ there were some $x_b \in S \cap [a, b]$ then $\bigwedge (x_b | b \in B) \in S$ since $S$ is a complete sublattice. On the other hand, $a \leq \bigwedge (x_b | b \in B) \leq \bigwedge B = a$, that is, $\bigwedge (x_b | b \in B) = a \notin S$, which is a contradiction. Thus, we can conclude that there is some $b \in B$ such that $S \cap [a, b] = \emptyset$.

**Lemma 2.** If $L$ is a distributive lattice, $a \in J(L)$, $b \in M(L)$, and $a \leq b$, then $L - [a, b]$ is a sublattice of $L$.

**Proof.** Let us assume that there exist $x, y \in L - [a, b]$ such that $x \vee y \in [a, b]$. Since $a \in J(L)$ and $L$ is distributive we get that $a \leq x$ or $a \leq y$, so that either $x \in [a, b]$ or $y \in [a, b]$, contradicting our choice.

We can now give the basic characterization [1].

**Theorem 3.** If $L$ is a finite distributive lattice and $M$ is a maximal proper sublattice, then there exist $a \in J(L)$, $b \in M(L)$, $a \leq b$, such that (i) $L - M = [a, b]$, (ii) $(a, b) \subseteq L - J(L)$ and (iii) $(a, b) \subseteq L - M(L)$.

**Proof.** (i) By Lemma 1 there exist $a \in J(L)$, $b \in M(L)$, $a \leq b$, such that $M \cap [a, b] = \emptyset$, that is, $M \subseteq L - [a, b] \subseteq L$. By Lemma 2, $L - [a, b]$ is a sublattice so that by the maximality of $M$, $M = L - [a, b]$. 
(ii) If there were some \( c \in (a, b] \) such that \( c \in J(L) \), then in view of Lemma 2, \( M = L - [a, b] \subseteq L - [c, b] \subseteq L \) is a chain of sublattices contradicting the maximality of \( M \). (iii) follows dually.

We are now ready to prove Theorems 1 and 2.

3. The Boolean case. Suppose \( M \) is a maximal proper sublattice of the Boolean lattice \( 2^n \). Without loss of generality we may take \( n \geq 3 \) in which case it must be that \( 0, 1 \in M \). By Theorem 3 there exist \( a \in J(L) \), \( b \in M(L) \) such that \( L - M = [a, b] \). But then \( a \) must be an atom and \( b \) a coatom, so that \( [a, b] \simeq 2^{n-2} \), and this, of course, implies that \( |M| = \frac{1}{2} |2^n| \).

Finally, if \( a' \) is the complement of \( a \), then \([0, a'] \cup [a', 1] \subseteq M \), since \( 0, 1 \in M \) and \( L - M \) is a convex sublattice of \( L \). Since all maximal chains have the same order, \( l(M) = l(L) \) which completes the proof of Theorem 1.

4. The distributive case. The proof of Theorem 2 amounts essentially to an enumeration of sufficiently many joins and meets of subsets of \( J(L) \) and \( M(L) \), respectively.

We can certainly assume that \( |L - M| \geq 2 \) so that by the maximality of \( M \), \( 0, 1 \in M \) and \( L - M \subseteq L - (J(L) \cap M(L)) \). By Theorem 3 there exist \( a \in J(L) \), \( b \in M(L) \), \( a \leq b \), such that \( L - M = [a, b] \).

Now, \( L \) is finite, so in particular, for every \( x \in (a, b] \) there exists \( A \subseteq J(L) \) such that \( x = \bigvee A \). If \( A \subseteq M \) then \( \bigvee A \in M \) since \( M \) is a finite sublattice. Therefore, \( A \cap [a, b] \neq \varnothing \), and, in fact, by Theorem 3, \( J(L) \cap [a, b] = \{a\} \), so that \( a \in A \). This shows that for every \( x \in (a, b) \) we can select a subset \( f(x) \in J(L) \cap M \) such that \( x = a \vee f(x) \), where \( \bigvee f(x) \in M \); in fact, \( \bigvee f(x) = M - \{0, 1\} \). Moreover, it is immediate that if \( x, y \in (a, b) \), \( \bigvee f(x) = \bigvee f(y) \) if and only if \( x = y \). Thus, we can conclude that

\[
\{\bigvee f(x) \mid x \in (a, b]\} \subseteq M - \{0, 1\}
\]

and

\[
|\{\bigvee f(x) \mid x \in (a, b]\}| = |(a, b]| = |L - M| - 1.
\]

Dually, we get a choice function \( g \) from \([a, b)\) into the set of subsets of \( M(L) \) such that \( y = b \wedge g(y) \), for every \( y \in [a, b) \), \( \{\wedge g(y) \mid y \in [a, b)\} \subseteq M - \{0, 1\} \), and \( |\{\wedge g(y) \mid y \in [a, b]\}| = |L - M| - 1 \).

We now claim that \( \{\bigvee f(x) \mid x \in (a, b]\} \cap \{\wedge g(y) \mid y \in [a, b]\} = \varnothing \). In fact, suppose there were suitable \( x, y \in [a, b) \) such that \( \bigvee f(x) = \wedge g(y) \). Clearly, \( b \geq x = a \vee f(x) \geq \bigvee f(x) \) so that \( \bigvee f(x) = b \vee f(x) = b \wedge g(y) = y \), which is impossible since \( \bigvee f(x) \in M \) while \( y \in L - M \).

It now follows that \( |L| \geq |L - M| + |\{\bigvee f(x) \mid x \in (a, b]\}| + |\{\wedge g(y) \mid y \in [a, b]\}| + |\{0, 1\}| = 3|L - M| \), from which (i) is an immediate consequence.

To establish (ii) take some \( c \in L \) such that \( b > c \geq \bigvee f(b) \) (\( b \) covers \( c \)). If \( c \in [a, b] \) then \( c \geq \bigvee f(b) \) and \( c \geq a \) together imply that \( b > c \geq a \vee f(b) = b \), which is impossible. Thus, \( c \in M \) and by the convexity of \( L - M \),
[0, c] \subseteq M. Finally, taking a maximal chain in [0, c], adjoining to it a maximal chain in (b, 1] and recalling that in distributive lattices all maximal chains have the same order, we can conclude that \( l(M) \geq l(L) - 1 \), completing the proof of Theorem 2.

**BIBLIOGRAPHY**


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