WHEN \((D[[X]])_{P[[X]]}\) IS A VALUATION RING

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Abstract. Let \(D\) be an integral domain with identity and let \(K\) denote the quotient field of \(D\). If \(P\) is a prime ideal of \(D\) denote by \(D[[X]]\) the prime ideal of \(D[[X]]\) consisting of all those formal power series each of whose coefficients belongs to \(P\). In this paper the following question is considered: When is \((D[[X]])_{P[[X]]}\) a valuation ring? Our main theorem states that if \((D[[X]])_{P[[X]]}\) is a valuation ring, then \(D_P\) must be a rank one discrete valuation ring. Moreover, we show that if \(D_P\) is a rank one discrete valuation ring and if \(PD[[X]] = P[[X]]\), then \((D[[X]])_{P[[X]]}\) is a valuation ring. We also give an example to show that \((D[[X]])_{P[[X]]}\) need not be a valuation ring when \(D_P\) is rank one discrete.

1. Main theorem. Let \(V\) be a rank one valuation ring with quotient field \(K\) and let \(M\) denote the maximal ideal of \(V\). If \(v\) is a valuation on \(K\) associated with \(V\), then we may take the value group of \(v\) to be a subgroup of the additive group of real numbers. For \(f = \sum_{i=0}^{\infty} f_i X^i\), define \(v^*(f) = \inf_{v(f) \leq v(g)} v(f_i)\). Equipped with this terminology, we have

**Lemma 1.** \(v^*\) is a valuation on \(V[[X]]\). Moreover, \(MV[[X]]\) is a prime ideal of \(V[[X]]\) and \((V[[X]])_{V^*[[X]]}\) is the valuation ring associated with \(v^*\).

**Proof.** Let \(f = \sum_{i=0}^{\infty} f_i X^i\), \(g = \sum_{i=0}^{\infty} g_i X^i \in V[[X]]\). It is straightforward to verify that \(v^*(f+g) \leq \min\{v^*(f), v^*(g)\}\). To show that \(v^*(fg) = v^*(f)+v^*(g)\), it suffices to show that \(v^*(fg) \geq v^*(f)+v^*(g)\). We first consider the case in which there exist coefficients \(f_i\) and \(g_s\) of \(f\) and \(g\) respectively, with the property that \(v^*(f) = v(f_i)\) and \(v^*(g) = v(g_s)\). If \(r\) and \(s\) are minimal with this property, then

\[
v \left( \sum_{i+j=r+s} f_i g_s \right) = v^*(f) + v^*(g)
\]

and since \((\sum_{i+j=r+s} f_i g_s)\) is the coefficient of \(X^{r+s}\) in \(fg\), it follows that \(v^*(fg) \geq v^*(f) + v^*(g)\). In particular, if \(v\) is a discrete valuation, this shows that \(v^*\) is a valuation. Assuming now that \(v\) is a nondiscrete valuation, let \(f, g\) be arbitrary elements of \(V[[X]]\) and let \(m \in M\setminus\{0\}\). If \(f' = \sum_{i=0}^{\infty} f_i m^i x^i\),
then either \( f' \) is a polynomial or \( \lim_{n \to \infty} v(f_n m^n) = \infty \). In either case, there exists an integer \( s \) such that \( v^*(f') = v(f m^s) \). Consequently, if we also set \( g' = \sum_{i=0}^{\infty} g_i m^i x^i \), then by the case previously handled, \( v^*(f' g') \leq v^*(f') + v^*(g') \). But

\[
v^*(fg) = \inf_{v(m) > 0} \{ v^*(f' g') \} \leq \inf_{v(m) > 0} \{ v^*(f') + v^*(g') \} = v^*(f) + v^*(g).
\]

Thus, \( v^* \) is a valuation. Since \( MV[[X]] = \{ f \in V[[X]] | v^*(f) > 0 \} \), it is the center of the valuation ring associated with \( v^* \), and therefore, is a prime ideal.

We now prove that \( (V[[X]])_{MV[[X]]} \) is the valuation ring associated with the valuation \( v^* \). Since \( V \) is a rank one valuation ring, \( V[[X]] \), and hence \( (V[[X]])_{MV[[X]]} \), are integrally closed. Thus, it suffices to show that \( v^* \) is the unique valuation on \( V[[X]] \) centered on \( MV[[X]] \) [2, p. 215]. Let \( w \) be such a valuation. If \( v' \) is the restriction of \( w \) to \( K \), then since \( V \) is the valuation ring associated with \( v' \), we may assume that \( v' = v \). If \( f \in V[[X]] - MV[[X]] \), then \( w(f) = 0 = v^*(f) \). If \( f \in MV[[X]] \) and if \( v^*(f) \) is an element of the value group of \( v \), then there exists \( m \in M \) such that \( v^*(f) = v(m) \) and hence there exists \( f' \in V[[X]] - MV[[X]] \) such that \( f = mf' \). Consequently, \( w(f) = v(m) + w(f') = v(m) + v^*(f) \). Now suppose that \( f \in MV[[X]] \) and that \( v^*(f) = \eta \) is not an element of the value group of \( v \). For \( \varepsilon > 0 \), let \( m_\varepsilon \in M \) be such that \( v(m_\varepsilon) = \eta_\varepsilon \), where \( 0 < \eta - \eta_\varepsilon < \varepsilon \). Since \( v^*(f) > \eta \), there exists \( f_\varepsilon \in V[[X]] \) such that \( f = mf_\varepsilon \). Thus, \( w(f) = w(f m_\varepsilon) = w(f_\varepsilon) + v(m_\varepsilon) > \eta_\varepsilon \) and it follows that \( w(f) \geq \eta = v^*(f) \). Now, let \( \alpha \) be an element of the value group of \( v \), \( \alpha > \eta \), and let \( g \in V[[X]] \) be such that \( v^*(g) = \alpha - \eta \). Then \( v^*(fg) = \alpha \), so by a previous case, \( v^*(f) + v^*(g) = v^*(fg) = w(f) = w(g) + w(f) \). Since \( w(f) \geq v^*(f) \) and \( w(g) \geq v^*(g) \), it follows that \( v^*(f) = w(f) \). Therefore, \( v^* \) and \( w \) agree on each element of \( V[[X]] \).

In the above proof, the argument that \( v^* \) is a valuation was shown to the authors by Marius Van der Put.

We now state our main result.

**Theorem 1.** Let \( P \) be a prime ideal of the integral domain \( D \). If \( (D[[X]])_{MV[[X]]} \) is a valuation ring, then \( D_P \) is a rank one discrete valuation ring. Moreover, \( (D[[X]])_{MV[[X]]} \) is rank one discrete.

**Proof.** Since \( D_P = (D[[X]])_{MV[[X]]} \cap K \), \( D_P \) is a valuation ring. If the height of \( P \) is greater than one, then the rank of \( D_P \) is greater than one. Therefore, there exists a prime ideal \( Q \) of \( D \) such that \( QD_P < PD_P \) and such that \( QD_P = \bigcap_{a=1}^{\infty} a^e D_P \) for some \( a \in P - Q \). We show that \( (D[[X]])_{Q[[X]]} \) is not a valuation ring by exhibiting an element \( \xi(X) \in \text{q.f.}(D[[X]]) \) such that neither \( \xi(X) \) nor \( (\xi(X))^{-1} \) belongs to \( (D[[X]])_{Q[[X]]} \).
Let \( q \in Q - (0) \). Then \( q \in \bigcap_{n=1}^{\infty} a^nD_P \), so that for each positive integer \( i \), there exist \( r_i, s_i \in D, s_i \notin P \) such that \( q = (r_i/s_i)a^i \). Set \( r_n = \sum_{j=0}^{\infty} r_j \) and let \( \xi(X) = \sum_{j=0}^{\infty} (s_j/a^j)X^j \). Then \( q \xi(X) = \sum_{j=0}^{\infty} (qs_j/a^j)X^j = \sum_{j=0}^{\infty} r_jX^j \in D[[X]] \) and therefore, \( \xi(X) \in \text{q.f.}(D[[X]]) \). Suppose that \( \xi(X) = h(X)/g(X) \), \( h(X), g(X) \in D[[X]] \), \( h(X) = \sum_{j=0}^{\infty} h_jX^j \), \( g(X) = \sum_{j=0}^{\infty} g_jX^j \). We show that \( h(X) \) and \( g(X) \) must be elements of \( Q[[X]] \). \( \xi(X)g(X) = h(X) \), so that for each positive integer \( k \) we have that \( \sum_{i+j=k} (g_is_{ij})/a^{ij} = h_k \) and hence that

\[
 g_0 s_{v_k} = a^{v_k} \left( h_k - \sum_{i+j=k+1} (g_is_{ij}/a^{ij}) \right) = a^{v_k} h_k = \sum_{i+j=k+1} a^{v_k-v} g_is_{v_i} \in a^kD. 
\]

Thus, \( g_0 \in a^kD_P \) for each positive integer \( k \) and it follows that \( g_0 \in (\bigcap_{n=1}^{\infty} a^nD_P) \cap D = Q \). Suppose that we have shown that \( g_i \in Q \) for \( 0 \leq i < n \). For an arbitrary positive integer \( k \), consider the equation \( \sum_{i+j=n+k} (g_is_{ij})/a^{ij} = h_{n+k} \). We see from this that

\[
 g_n s_{v_k} = a^{v_k} \left( h_{n+k} - \sum_{i+j=n+k+1} (g_is_{ij}/a^{ij}) - \sum_{i+j=n+k+1} (g_is_{ij}/a^{ij}) \right) = a^{v_k} h_{n+k} - \sum_{i+j=n+k+1} a^{v_k-v} g_is_{v_i} - a^{v_k} \sum_{i+j=n+k+1} (g_is_{ij}/a^{ij}). 
\]

For \( 0 \leq i < n \), \( g_i \in Q \). Therefore, \( g_is_{v_i} \in (a^{i+1})D_P \) and hence, \( (g_is_{v_i})/a^{ij} \in a^kD_P \). It now follows that \( g_n s_{v_k} \in a^kD_P \). Thus, \( g_n \in (\bigcap_{n=1}^{\infty} a^nD_P) \cap D = Q \). By induction, \( g(X) \in Q[[X]] \). To see that \( h(X) \in Q[[X]] \), we note that for any nonnegative integer \( n \), \( h_n = \sum_{i+j=n} (g_is_{ij})/a^{ij} \). From a repetition of the above argument, we conclude that \( h_n \in Q \). It follows that neither \( \xi(X) \) nor \( [\xi(X)]^{-1} \) belongs to \( (D[[X]])_{Q[[X]]} \). This contradicts the fact that \( (D[[X]])_{Q[[X]]} \) is a valuation ring and hence we must have that \( D_P \) is a rank one valuation ring.

Suppose now that \( D_P \) is nondiscrete and let \( v \) be a valuation on \( K \) associated with \( D_P \). For notational purposes, put \( D_P = V, PD_P = M \) and let \( v^* \) denote the extension of \( v \) to \( V[[X]] \) as defined in Lemma 1. Choose \( m \in P - (0) \) and \( f(X) \in P[[X]] \). From \( f(X) \in P[[X]] \), we show that neither \( mf \) nor \( f/m \) belongs to \( (D[[X]])_{P[[X]]} \). If \( mf \neq g/h \), then \( mh = fg \in MV[[X]] \), a prime ideal by Lemma 1. Therefore, \( g \in MV[[X]] \cap D[[X]] \subseteq M[[X]] \cap D[[X]] = P[[X]] \). Now, \( v^*(g) = v^*(f) + v^*(g) = v^*(mh) = v(m) + v(h) = v(m) + v(h) = v(m) + v(h) \). Thus, there exists \( g' \in V[[X]] \) such that \( g = mg' \) and hence, \( h = fg' \in P[[X]] \). We conclude that \( D_P \) is discrete. Furthermore, since \( D_P \) is discrete, \( MV[[X]] = M[[X]] \). By Lemma 1, \( (V[[X]])_{M[[X]]} \) is a rank one discrete valuation ring and hence \( (V[[X]])_{M[[X]]} \cap \text{q.f.}(D[[X]]) \) is
a rank one discrete valuation ring centered on $P[[X]]$. But $(D[[X]])_{P[[X]]}$ is a valuation ring and therefore equals $(V[[X]])_{M[[X]]} \cap q.f.(D[[X]])$.

Thus, in order that $(D[[X]])_{P[[X]]}$ be a valuation ring it is necessary that $D_x$ be a rank one discrete valuation ring. However, this condition is not sufficient as we shall presently show. We require some preliminaries.

The proof of the following proposition for the case when $R$ is a rank one nondiscrete valuation ring was communicated to the authors by David Fields. The proof given here is essentially a modification of his argument.

**Proposition 1.** Let $R$ be a commutative ring with identity and let $M$ be a maximal ideal of $R$. If $Q$ is a prime ideal of $R[[X]]$ containing $MR[[X]]$, then $Q \supseteq M[[X]]$ or $Q = M + (X)$. Therefore, if $M[[X]] \supseteq \sqrt{(MR[[X]])}$, then there exists a prime ideal $Q$ of $R[[X]]$ such that $MR[[X]]^Q \subseteq Q 
subseteq M[[X]]$.

**Proof.** The set $Q' = \{ a \in R | f(0) = a \text{ for some } f \in Q \}$ is an ideal of $R$ and since $Q \supseteq MR[[X]]$, $Q' \supseteq M$. Thus, $Q' = M$. Suppose that $Q \not\subseteq M[[X]]$ and let $f = \sum_{i=0}^\infty f_i X^i \in Q - M[[X]]$. If $t$ is minimal among those positive integers $j$ such that $f_j \not\in M$, then $g = \sum_{i=0}^t f_i X^i \in MR[[X]] \subseteq Q$ and hence $f - g = \sum_{i=t+1}^\infty f_i X^i \in Q$. Since $f_0 = Q' = M$, $t \neq 0$. Thus, $f - g = X^t(\sum_{i=t+1}^\infty f_i X^{i-t}) \in Q$ and $\sum_{i=t}^\infty f_i X^{i-t} \not\in Q$ since $f_i \not\in M$. Therefore, $X \in Q$ and $Q \supseteq M + (X)$, maximal.

The second assertion requires no proof.

Let $R$ be a commutative ring with identity. If $f = \sum_{i=0}^\infty f_i X^i \in R[[X]]$, then we denote by $A_f$ the ideal of $R$ generated by the coefficients of $f$.

**Lemma 2.** Let $R$ be a commutative ring with identity. If $f = \sum_{i=0}^\infty f_i X^i \in R[[X]]$, then $s/(A_f) = \sqrt{(A_f^2)}$ for each positive integer $n$.

**Proof.** Since $A_f \supseteq A_{f^2} \supseteq \cdots$ is a decreasing sequence of ideals of $R$, it follows that for a positive integer $n$, $\sqrt{(A_f)} \supseteq \sqrt{(A_{f^n})} \supseteq \sqrt{(A_{f^{2n}})}$. Thus, it suffices to prove that $\sqrt{(A_f)} = \sqrt{(A_{f^{2^n}})}$ for each positive integer $n$. To do this we use induction on $n$. Set $Q = \sqrt{(A_f)}$. Then $Q \supseteq A_{f^n}$ and therefore $Q \supseteq \sqrt{(A_{f^n})}$. Let $P$ be a prime ideal of $R$ such that $P \supseteq A_{f^n}$. If $P \not\supseteq Q$, then $P \not\supseteq A_f$ since $Q = \sqrt{(A_f)}$. Let $t$ be the smallest positive integer $i$ such that $f_i \not\in P$. Then $\sum_{i=0}^{t-1} 2 f_i f_{2t-i} + f_t^2$ is the coefficient of $X^{2t}$ in $f^2$ and for $0 \leq i \leq t-1$, $f_i \in P$. Thus, $\sum_{i=0}^{t-1} 2 f_i f_{2t-i} \in P$ and it follows that $f_t^2$, hence $f_t$, belongs to $P$. This contradiction shows that $P \supseteq Q$ and therefore $Q = \sqrt{(A_{f^n})}$. This proves the assertion in case $n = 1$. Assume the result to be true for $m$. By the case just proved, $\sqrt{(A_{f^{2n+1}})} = Q$ since $(f^{2^n})^2 = f^{2^{n+1}}$.

**Corollary 1.** Let $R$ be a commutative ring with identity and let $P$ be a prime ideal of $R$. If $f = \sum_{i=0}^\infty q_i X^i \in P[[X]]$ is such that $\sqrt{(A_f)} = P$ and if $P$ is
not the radical of a finitely generated ideal of $R$, then $f \notin (PR[[X]])$. In particular, $P[[X]] \supset (PR[[X]])$.

**Proof.** It is straightforward to show that if $g \in P[[X]]$ is such that $P = \sqrt{(A_0)}$, then $g \notin PR[[X]]$. Thus, by Lemma 2, no power of $f$ belongs to $PR[[X]]$.

**Example.** We now give an example of a domain $D$ containing a prime ideal $P$ such that $D_P$ is a rank one discrete valuation ring but such that $(D[[X]])_{P[[X]]}$ is not a valuation ring. Let $D$ be a countable almost Dedekind domain which is not Dedekind. (See [2, p. 585].) Let $M$ be a maximal ideal of $D$ which is not finitely generated and let $\{m_i\}_{i=0}^{\infty}$ be a countable basis for the ideal $M$. Since $M$ is not finitely generated it is easy to see that $M$ is not the radical of a finitely generated ideal. It follows from Corollary 1 that $f = \sum_{i=0}^{\infty} m_i X^i \in M[[X]] - MD[[X]]$ and therefore we have that $M[[X]] \supset (MD[[X]])$. By Proposition 1, $M[[X]]$ is not a height one prime of $D[[X]]$, but by Theorem 1, if $(D[[X]])_{M[[X]]}$ is a valuation ring, it is rank one discrete. Thus, $D$ and $M$ provide the desired example.

We see from the above example that if $P$ is a prime ideal of a domain $D$ such that $D_P$ is a rank one discrete valuation ring, then it need not be true that $(D[[X]])_{P[[X]]}$ is a valuation ring. However, if $D_P$ is a rank one discrete valuation ring and if $PD[[X]] = P[[X]]$, then $(D[[X]])_{P[[X]]}$ is a valuation ring. This is a consequence of

**Theorem 2.** Suppose that $P$ is a prime ideal of an integral domain $D$ such that $D_P$ is a rank one discrete valuation ring. If $PD[[X]]$ is a prime ideal of $D[[X]]$, then $(D[[X]])_{PD[[X]]}$ is a rank one discrete valuation ring.

**Proof.** Let $PD_P = qD_P$, where $q \in P$ and let $f(X) = \sum_{i=1}^{n} q_i h_i(X) \in PD[[X]]$, where $q_i \in P$, $h_i(X) \in D[[X]]$. Since $q_i D_P \subseteq q D_P$ for $1 \leq i \leq n$, it follows that there exists $f^{(i)}(X) \in D[[X]]$ and an $s_i \in D - P$ such that $f(X) = q f^{(i)}(X)/s_i$. If $f^{(i)}(X) \in PD[[X]]$, then we may repeat the procedure. If this process failed to terminate, then for each positive integer $n$, we would obtain $f^{(n)}(X) \in D[[X]]$ and $s_n \in D - P$ such that $f(X) = q^n f^{(n)}(X)/s_n$. Therefore,

$$f(X) \in \bigcap_{n=1}^{\infty} q^n(D_P[[X]])_{PD_P[[X]]} \cap q.f.(D[[X]]) = (0).$$

Thus, if $f(X) \neq 0$, then there exists a positive integer $n$ such that $f(X) = q^n h(X)/s$, where $h(X) \in D[[X]] - PD[[X]]$ and $s \in D - P$. To prove that $(D[[X]])_{PD[[X]]}$ is a valuation ring, it suffices to show that given $f(X)$, $g(X) \in PD[[X]]$, either $f(X)/g(X)$ or $g(X)/f(X) \in (D[[X]])_{PD[[X]]}$, but this is clear from the above argument. In fact, we have shown that each element
of \((D[[X]])_{PD([X])}\) is of the form \(uq^r\), where \(u\) is a unit of \((D[[X]])_{PD([X])}\) and \(r\) is a nonnegative integer. Therefore, \((D[[X]])_{PD([X])}\) is rank one discrete.

We do not know an example of a domain \(D\) containing a prime ideal \(P\) satisfying the hypotheses of Theorem 2 for which \(PD[[X]]\) is a valuation ring, but we conjecture that such examples do exist.

Let \(P\) be a prime ideal of \(D\) with the property that \(DP\) is a rank one discrete valuation ring. We see from Theorem 2 that if \(PD[[X]]\) is a valuation ring, then \((D[[X]])_{PD([X])}\) is a valuation ring. However, \((D[[X]])_{PD([X])}\) can be a valuation ring even though \(PD[[X]]\) is not. Namely, using the construction of Eakin and Heinzer in [1, p. 1264] one can exhibit a Krull domain \(R\) having a minimal prime ideal \(Q\) such that \(Q\) is countably generated, but not finitely generated. By [3, p. 386], \(QR[[X]]\) is a valuation ring \([3, p. 380]\). Thus, while the condition that \(PD[[X]]\) is sufficient to guarantee that \((D[[X]])_{PD([X])}\) is a valuation ring, it is not necessary, and consequently any condition which is both necessary and sufficient must be weaker. Therefore, the following result, while not entirely satisfactory, seems at present to be the best characterization of when \((D[[X]])_{PD([X])}\) is a valuation ring.

**Theorem 3.** Let \(P\) be a prime ideal of an integral domain \(D\) with the property that \(DP\) is a rank one discrete valuation ring. The following conditions are equivalent:

1. \((D[[X]])_{PD([X])}\) is a valuation ring.
2. \(P(D[[X]])_{PD([X])}=P[[X]](D[[X]])_{PD([X])}\).
3. \(D_P[[X]]\cap q.f.(D[[X]]) (D[[X]])_{PD([X])}\).

**Proof.** Throughout, assume that \(PD=QD_P\) and that \(V=\{(D_P[[X]])_{PD([X])}\cap q.f.(D[[X]])\neq (0)\).

1. \(\Rightarrow\) (2): \(P([[X]])(D[[X]])_{PD([X])}=q(D[[X]])_{PD([X])}\subseteq PD([[X]])_{PD([X])}\).

2. \(\Rightarrow\) (1): If \(f(X)\in P([[X]])\), then by modifying the proof of Theorem 2, we see that \(f(X)=q^n\xi\), where \(\xi\) is a unit of \((D[[X]])_{PD([X])}\). Therefore, if \(\eta=f(X)/g(X)\in q.f.(D[[X]])\), where both \(f(X), g(X)\) belong to \(P([[X]])\), then \(\eta\) or \(\eta^{-1}\) belongs to \((D[[X]])_{PD([X])}\), depending upon which of \(f(X)\) and \(g(X)\) is in the higher power of \(q(D[[X]])_{PD([X])}\).

3. (3) \(\Rightarrow\) (1): \((D([[X]])_{PD([X])}\), being a quotient ring of the Krull domain \(D_P[[X]]\cap q.f.(D[[X]])\), is itself a Krull domain and \(V\), being an essential valuation overring of \(D_P[[X]]\cap q.f.(D[[X]])\) containing \((D([[X]])_{PD([X])}\), is essential for \((D([[X]])_{PD([X])}\). Since \(V\) is centered on \(P([[X]])\), \(V=(D([[X]])_{PD([X])}\).

(1) \(\Rightarrow\) (3): \(V=(D([[X]])_{PD([X])}\subseteq D_P[[X]]\cap q.f.(D([[X]])).\)
References


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