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UNIFORM EQUICONTINUITY OF QUASICONFORMAL MAPPINGS[1]

RAIMO NÄKKI AND BRUCE PALKA

ABSTRACT. Necessary and sufficient conditions are given for a family of K-quasiconformal mappings of a fixed domain to be uniformly equicontinuous in that domain, provided the domain has a "regular" boundary. Applications to uniformly convergent sequences of quasiconformal mappings are indicated.

1. **Introduction.** It is the object of this paper to consider the relationship between the uniform equicontinuity of a family of quasiconformal mappings and the problem of extending quasiconformal mappings continuously to the boundary of their domains of definition. It is shown that in many important cases the property of uniform equicontinuity and the possibility of extension are equivalent. Some results on the uniform convergence of sequences of quasiconformal mappings are also established. Many of our results are new, as well, for conformal mappings in the plane. Similar questions have been studied by Syčev [7].

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2. **Preliminaries.** We consider sets in $\bar{R}^n$, $n \geqq 2$, the Möbius space obtained by adding the point ∞ to Euclidean n-space R^n. Given a set E, we let ∂E and $\bar{E}$ denote the boundary and the closure of E in $\bar{R}^n$. As a metric in $\bar{R}^n$ we use the chordal metric q and we let $q(E)$ denote the diameter of the set E. A path in a domain D is a continuous mapping of a closed line interval into D and $\Delta(E, F: D)$ will denote the family of all paths in D which join the set E to the set F. The *modulus* of a family Δ of paths is designated by $M(\Delta)$. A homeomorphism f of a domain D is said to be *K-quasiconformal*, $1 \leqq K < \infty$, if $M(\Delta)/K \leqq M(f\Delta) \leqq KM(\Delta)$ for each family Δ of paths in D. A *quasiconformal* mapping is one which is K-quasiconformal for some K.

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2.1. *Equicontinuity and uniform equicontinuity.* A family $\mathscr{F}$ of mappings of a set E into $\bar{R}^n$ is said to be *equicontinuous* at a point $x \in E$ if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $q(f(x), f(y)) < \varepsilon$ whenever $f \in \mathscr{F}$ and $y \in E$, with $q(x, y) < \delta$. The family $\mathscr{F}$ is *uniformly equicontinuous* if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $q(f(x), f(y)) < \varepsilon$ whenever $f \in \mathscr{F}$ and $x, y \in E$, with $q(x, y) < \delta$.

Equicontinuity of n-dimensional quasiconformal mappings has been considered by Gehring [**1**], and, more recently, by Srebro [**6**] and Väisälä [**8**, §19].

2.2. *Quasiconformal collaredness.* In the study of the relationship between the uniform equicontinuity and the boundary extension of quasiconformal mappings, we will assume that at least one of the domains in question satisfies the following smoothness condition: Each boundary point of the domain has an arbitrarily small neighborhood such that the part of the neighborhood inside the domain is quasiconformally equivalent to a ball. A domain with this property is said to be *quasiconformally collared* on the boundary. Such domains have only finitely many boundary components and include, for example, bounded convex domains, polyhedra, and domains bounded by finitely many disjoint, smooth $(n-1)$-manifolds. A plane domain is quasiconformally collared on the boundary if and only if its boundary consists of a finite number of disjoint Jordan curves. (See [**2**] and Väisälä [**8**, §17].)

2.3. *Finite connectedness.* A domain D is said to be *finitely connected* on the boundary if each boundary point of D has arbitrarily small neighborhoods U such that $U \cap D$ consists of a finite number of components. (See [**2**] and Väisälä [**8**, §17].)

2.4. *Uniform domains.* A domain D is called a *uniform domain* if, for each $r > 0$, there is a $\delta > 0$ such that $M(\Delta(F, F^*: D)) \geqq \delta$ whenever F and F^* are connected subsets of D with $q(F) \geqq r$ and $q(F^*) \geqq r$. Domains D_i, $i \in I$, are said to be *equi-uniform domains* if, for $r > 0$, the modulus condition above is satisfied by each D_i with the same number δ.

Quasiconformally collared domains are uniform domains and uniform domains are finitely connected on the boundary. The converse relations are not true. However, a plane domain with finitely many boundary components is a uniform domain if and only if it is finitely connected on the boundary. (See [**3**, §6].)

3. **Uniform equicontinuity and boundary extension.** We first consider the case of fixed domains.

3.1. THEOREM. *Let $\mathscr{F}$ be a family of K-quasiconformal mappings of a domain $D \neq \bar{R}^n$ onto a domain D' and let either D or D' be quasiconformally collared on the boundary. Then $\mathscr{F}$ is uniformly equicontinuous if*

and only if each $f \in \mathscr{F}$ can be extended to a continuous mapping of $\bar{D}$ onto $\bar{D}'$ and $\inf_{\mathscr{F}} q(fA) > 0$ *for some continuum A in D.*

PROOF. Suppose first that $\mathscr{F}$ is uniformly equicontinuous. Then each $f \in \mathscr{F}$ is uniformly continuous and therefore extends to a continuous mapping of $\bar{D}$ onto $\bar{D}'$. Fix a continuum A in D. If the second assertion is false, there is a sequence (f_k) in $\mathscr{F}$ such that $q(f_k A) \to 0$. Since each f_k omits two fixed values, (f_k) is a normal family and we may therefore assume that $f_k \to f$ uniformly on compact subsets of D, where f is either a homeomorphism of D onto D' or a constant in $\partial D'$ (Väisälä [**8**, §§20–21]). The first case is impossible because $q(fA)=0$. The second case is impossible because, by the uniform equicontinuity of $\mathscr{F}$, f_k tends to f uniformly in D.

To prove the converse, it suffices to show that the extended mappings are equicontinuous at each point $b \in \partial D$, since $\mathscr{F}$ is equicontinuous in D by Väisälä [**8**, 19.3]. Fix such a point b. Then there is a decreasing sequence of neighborhoods U_k of b such that $U_k \cap D$ is connected and $\lim M(\Delta(A, U_k \cap D : D)) = 0$ (cf. [**2**, 3.1]). Since quasiconformally collared domains and their uniformly continuous images are uniform domains (cf. [**2**, 3.2]), D' is a uniform domain, and the K-quasiconformality of the mappings in $\mathscr{F}$, together with the condition $\inf_{\mathscr{F}} q(fA) > 0$, implies that $\lim_k \sup_{\mathscr{F}} q(f(U_k \cap D)) = 0$. The proof is complete.

In Theorem 3.1, if D has at least two boundary components, then $\inf q(fA) > 0$, $f \in \mathscr{F}$, for every continuum A in D (Väisälä [**8**, 21.14]). Thus we have:

3.2. THEOREM. *If one of the domains D and D' is quasiconformally collared on the boundary and has at least two boundary components, then the uniform equicontinuity of a family $\mathscr{F}$ of K-quasiconformal mappings of D onto D' is equivalent to the possibility of extending each $f \in \mathscr{F}$ continuously to $\bar{D}$.*

We next consider the case of variable domains.

3.3. THEOREM. *Let D be a domain which is quasiconformally collared on the boundary, let $\mathscr{F}$ be an equicontinuous family of K-quasiconformal mappings of D into $\bar{R}^n$, and let* $\inf q(fA) > 0$, *$f \in \mathscr{F}$, for some continuum A in D. Then the following statements are equivalent:*

(1) *$\mathscr{F}$ is uniformly equicontinuous.*

(2) *Each $f \in \mathscr{F}$ can be extended to a continuous mapping $\bar{f}$ of $\bar{D}$ into $\bar{R}^n$ and the family of restricted mappings $\bar{f}|\partial D$ is equicontinuous.*

(3) *The domains fD, $f \in \mathscr{F}$, are equi-uniform domains.*

PROOF. Obviously (1) implies (2), while (2) implies (1) by [**5**, Theorem 1]. (The set ∂D satisfies the regularity conditions assumed in [**5**, Theorem 1] by virtue of [**2**, §2] and Väisälä [**8**, §17].)

To prove the implication (1)$\Rightarrow$(3), fix $r>0$. By the uniform equicontinuity of $\mathscr{F}$, there is an $s>0$ such that $q(f^{-1}F)\geqq s$ for any $f\in\mathscr{F}$ and for any set F in fD with $q(F)\geqq r$. Since D is quasiconformally collared on the boundary, hence a uniform domain, it follows that

$$M(\Delta(F, F^*:fD)) \geqq M(\Delta(f^{-1}F, f^{-1}F^*: D))/K \geqq \delta/K$$

whenever $f\in\mathscr{F}$ and F and F^* are connected sets in fD with $q(F)\geqq r$ and $q(F^*)\geqq r$, where δ is a positive constant corresponding to the domain D and the number s in the definition of a uniform domain. This shows that the domains fD, $f\in\mathscr{F}$, are equi-uniform domains.

The implication (3)$\Rightarrow$(1) follows as in Theorem 3.1, because each $f\in\mathscr{F}$ can be extended to a continuous mapping of $\bar{D}$ by [**2**, 3.2].

As applications of Theorems 3.1–3.3 one obtains the following results for conformal mappings in R^2:

3.4. COROLLARY. *Let D and D' be bounded domains in R^2, one of whose boundaries consists of a finite number of disjoint Jordan curves, and let $\mathscr{F}$ be a family of conformal mappings of D onto D'. Then $\mathscr{F}$ is uniformly equicontinuous if and only if each $f\in\mathscr{F}$ can be extended to a continuous mapping of $\bar{D}$ onto $\bar{D}'$ and* $\inf|f'(P)|>0$, $f\in\mathscr{F}$, *for some point P in D. The condition* $\inf|f'(P)|>0$ *can be removed if D has at least two boundary components.*

PROOF. Since D and D' are bounded, the conditions "$\inf_{\mathscr{F}}|f'(P)|>0$ for some point P in D" and "$\inf_{\mathscr{F}} q(fA)>0$ for some continuum A in D" are each equivalent to the condition that $\mathscr{F}$ contains no sequence converging to a constant. Thus the corollary follows from Theorems 3.1 and 3.2.

3.5. COROLLARY. *Let $\mathscr{F}$ be a family of conformal mappings of the unit disk $B^2=\{x\in R^2:|x|<1\}$ into R^2 such that $f(0)=0$ and $0<a\leqq|f'(0)|\leqq b<\infty$ for each $f\in\mathscr{F}$. Then $\mathscr{F}$ is uniformly equicontinuous if and only if the domains fB^2, $f\in\mathscr{F}$, are equi-uniform domains.*

PROOF. Theorem 3.3 will apply once it is established that $\mathscr{F}$ is equicontinuous in B^2 and that $\inf_{\mathscr{F}} q(fA)>0$ for some continuum A in B^2. Since each $f\in\mathscr{F}$ omits the value ∞, $\mathscr{F}$ is equicontinuous at each point of B^2, except possibly at the origin (Väisälä [**8**, §19]). But since

$$a\,|x|/(1+|x|)^2 \leqq |f(x)| \leqq b\,|x|/(1-|x|)^2$$

for all $f\in\mathscr{F}$ and $x\in B^2$ by Koebe's inequalities, $\mathscr{F}$ is equicontinuous also at the origin. Moreover, any closed disk of radius r, $0<r<1$, with center at the origin can be chosen for the continuum A.

3.6. REMARKS. (1) The condition $\inf_{\mathscr{F}} q(fA)>0$ cannot be removed in Theorem 3.1. It may, however, be replaced by several other conditions, e.g. by the requirement that the set $\{f(P):f\in\mathscr{F}\}$ be contained in a compact subset of D' for some point P in D, or by the condition that $\mathscr{F}$ contains no sequence converging to a constant.

(2) There are many ways to describe the regularity of the boundary of a domain so that, in 3.1–3.5, each of the mappings in question can be extended to a continuous mapping between the closures. See [**2**].

(3) We note that in 3.1–3.5, the uniform equicontinuity of the family $\mathscr{F}$ is equivalent to the equicontinuity of the family of extended mappings.

4. **Uniform normality and uniform convergence.** We say that a family $\mathscr{F}$ of mappings in a domain D into $\bar{R}^n$ is *uniformly normal* if every sequence in $\mathscr{F}$ contains a subsequence which converges uniformly in D. By Ascoli's theorem, uniform equicontinuity implies uniform normality. Thus we obtain the following result directly from results in §3:

4.1. THEOREM. *If any of the conditions in* 3.1–3.5 *is satisfied, then the family* $\mathscr{F}$ *is uniformly normal.*

We conclude by considering convergent sequences of K-quasiconformal mappings. By Väisälä [**8**, §21], the limit mapping of such a sequence is either a constant, a mapping assuming exactly two values, or a homeomorphism. We are interested in the third case only. For fixed domains we have:

4.2. THEOREM. *Let D and D' be two domains, one of which is quasiconformally collared on the boundary, let (f_k) be a sequence of K-quasiconformal mappings of D onto D' converging pointwise to a homeomorphism f, and suppose that each f_k can be extended to a continuous mapping $\bar{f}_k$ of $\bar{D}$ onto $\bar{D}'$. Then f can be extended to a continuous mapping $\bar{f}$ of $\bar{D}$ onto $\bar{D}'$ and $\bar{f}_k\to\bar{f}$ uniformly on $\bar{D}$.*

PROOF. Since f is a K-quasiconformal mapping of D onto D' (Väisälä [**8**, §37]), f can be extended to a continuous mapping of $\bar{D}$ onto $\bar{D}'$ by [**2**, 3.3]. Since the family $(\bar{f}_k)$ is equicontinuous by Theorem 3.1 and Remark 3.6(3), Ascoli's theorem implies that $\bar{f}_k\to\bar{f}$ uniformly on $\bar{D}$.

In fact, uniform convergence occurs inside the domain in a fairly general situation, regardless of the boundary behavior of the mappings:

4.3. THEOREM. *Let D' be a uniform domain and let (f_k) be a sequence of K-quasiconformal mappings of a domain D onto D' converging pointwise to a homeomorphism f. Then (f_k) converges uniformly to f.*

PROOF. Suppose that the assertion is false. Then there is a $d>0$, a sequence of points $P_k\in D$, and a subsequence of (f_k), denoted again by

(f_k), such that $q(f_k(P_k), P_k') \geqq d$ for each k, where $P_k' = f(P_k)$. We may assume that (P_k') converges to some point P'. Since $f_k \to f$ uniformly on compact subsets of D (Väisälä [**8**, §21]), it follows that $P' \in \partial D'$.

Fix a continuum A in D, let $A' = fA$ and $r = \min\{d/2, q(A')/2\}$, and let δ be a positive constant corresponding to the domain D' and the number r in the definition of a uniform domain. Next pick a neighborhood U' of P' so that $q(U') < d/2$ and $M(\Delta(A', U' : D')) < \delta/K^2$. Since D' is finitely connected on the boundary, we may assume that $U' \cap D'$ contains only a finite number of components. Thus, by relabeling, we may further assume that all the points P_k' are contained in a single component, C', of $U' \cap D'$. Now since f is K-quasiconformal (Väisälä [**8**, §37]),

$$M(\Delta(A, C : D)) \leqq KM(\Delta(A', C' : D')) < \delta/K,$$

where $C = f^{-1}C'$. But if we fix an index k so that $f_k C$ meets C' and $q(f_k A) \geqq q(A')/2$, which can be done because $f_k \to f$ uniformly on compact subsets of D, we have

$$M(\Delta(f_k A, f_k C : D')) \geqq \delta$$

by the fact that $q(f_k A) \geqq r$ and $q(f_k C) \geqq d/2 \geqq r$. This contradicts the K-quasiconformality of the mapping f_k.

4.4. Corollary. *Let D' be a plane domain with finitely many boundary components and let D' be finitely connected on the boundary. Then every sequence (f_k) of conformal (K-quasiconformal) mappings of a domain D onto D' which converges pointwise to a homeomorphism f converges uniformly to f.*

For variable domains we have:

4.5. Theorem. *Let $D \neq \bar{R}^n$ be a domain which is quasiconformally collared on the boundary, let (f_k) be a sequence of K-quasiconformal mappings of D into $\bar{R}^n$ converging pointwise to a homeomorphism f, and suppose that each f_k can be extended to a continuous mapping $\bar{f}_k$ of $\bar{D}$ into $\bar{R}^n$. Then the following statements are equivalent:*

(1) *$(\bar{f}_k)$ converges uniformly on $\bar{D}$.*

(2) *$(\bar{f}_k | \partial D)$ converges uniformly on ∂D.*

(3) *The domains $f_k D$ are equi-uniform domains.*

Proof. Obviously (1) implies (2). To prove that (2) implies (3), observe that, by Ascoli's theorem, $(\bar{f}_k | \partial D)$ is equicontinuous and since $f_k \to f$ uniformly on compact subsets of D (Väisälä [**8**, §21]), $(\bar{f}_k | D)$ is also equicontinuous. Hence $(\bar{f}_k)$ is equicontinuous on $\bar{D}$ by Theorem 3.3 and (3) follows. Finally, Ascoli's theorem, together with Theorem 3.3 and Remark 3.6(3), proves the implication (3)$\Rightarrow$(1).

4.6. REMARKS. (1) In Theorem 4.5, the uniform convergence of (f_k) on compact subsets of D does not, in general, imply that the limit mapping f extends to a continuous mapping of $\bar{D}$, even if each f_k does. In the case of the uniform convergence of (f_k) on all of D, of course, f will have an extension to $\bar{D}$, providing each f_k does.

(2) For further results on the uniform convergence of quasiconformal mappings, see [4] and [5].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HELSINKI, FINLAND

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912