INNER PRODUCTS CHARACTERIZED
BY DIFFERENCE EQUATIONS

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ABSTRACT. A normed linear space $X$ is an inner product space iff, for some integer $k \geq 3$, \( \sum_{t=0}^{k} \binom{k}{t} (-1)^t \|a + tb\|^2 = 0 \) for all $a$ and $b$ in $X$.

THEOREM. If $X$ is a linear space with norm $\| \cdot \|$ and, for some integer $k \geq 3$,
\[
\sum_{t=0}^{k} \binom{k}{t} (-1)^t \|a + tb\|^2 = 0
\]
for all $a, b \in X$ then $\sum_{t=0}^{k} \binom{k}{t} (-1)^t \|a + tb\|^2 = 0$ for every integer $k \geq 3$ and $\| \cdot \|$ is induced by an inner product on $X$.

DEFINITION. Suppose $X$ is a normed linear space and $k$ and $n$ are non-negative integers. Let
\[
D^n_k(a, b) = \sum_{t=0}^{k} \binom{k}{t} (-1)^t \|a + (t + n)b\|^2
\]
where $a$ and $b$ are in $X$ and $\| \cdot \|$ denotes the norm on $X$.

PROOF. Suppose $k$ is an integer greater than 2 and $D^n_k(a, b) = 0$ for all $a$ and $b$. Then $D^n_k(a, b) = 0$ if $n$ is a nonnegative integer; moreover, $D^n_k(a, b) = D^{n+1}_{k-1}(a, b) - D^n_{k-1}(a, b)$ and hence $D^n_{k-1}(a, b) = D^0_{k-1}(a, b)$.

Suppose $m$ is a positive integer not exceeding $k$ then by iteration we have that
\[
(1) \quad D^n_{k-m}(a, b) = \sum_{t=0}^{m-1} \binom{n}{t} D^0_{k-m+t}(a, b) \quad \text{for } n = 0, 1, 2, \cdots,
\]
and hence
\[
(2) \quad D^n_0(a, b) = \sum_{t=0}^{k-1} \binom{n}{t} D^0_t(a, b) \quad \text{for } n = 0, 1, 2, \cdots.
\]

Recall that $D^n_0(a, b) = \|a + nb\|^2$. Hence it follows that
\[
(3) \quad \|(1/n)a + b\|^2 = \sum_{t=0}^{k-1} \binom{n}{t} D^0_t(a, b)/n^2.
\]
We have then that
\[
\lim_{n \to \infty} \| (1/n)a + b \|^2 = \| b \|^2
\]
\[
= \lim_{n \to \infty} \frac{1}{n^2} \sum_{i=0}^{k-1} \left( \frac{n}{i} \right) D_i^0(a, b)
\]
\[
= \lim_{n \to \infty} \left[ D_0^0(a, b)/n^2 + \left( \left( \frac{n}{1} \right)/n^2 \right) D_1^0(a, b) + \left( \left( \frac{n}{2} \right)/n^2 \right) D_2^0(a, b)
\right.
\]
\[
+ \left( \left( \frac{n}{3} \right)/n^2 \right) D_3^0(a, b) + \cdots + \left( \left( \frac{n}{k-1} \right)/n^2 \right) D_{k-1}^0(a, b) \Bigg].
\]

In order that this limit exist it is necessary that \( D_0^0(a, b) = 0 \) if \( 3 \leq l \leq k - 1 \). Hence the limit is \( \frac{1}{2} D_0^0(a, b) \) if \( k \geq 3 \). Therefore
\[
\| b \|^2 = \frac{1}{2} \left[ \| a + 2b \|^2 - 2\| a + ab \|^2 + \| a \|^2 \right]
\]
for all \( a \) and \( b \) in \( X \), which is a simple reformulation of the parallelogram law.

Hence the space is an inner product space from which it is easy to establish that \( D_k^0(a, b) = 0 \) if \( k \) is any integer greater than 2.

**Reference**


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