ON THE EXISTENCE OF REGULAR APPROXIMATE DIFFERENTIALS

ALBERT FADELL

Abstract. We prove that for continuous real-valued functions on an open set in n-space, a sufficient condition for the existence a.e. of a regular approximate differential is that the functions have an ordinary total differential a.e. with respect to all but one variable.

1. Introduction. T. Radó ([4], [5]) and R. Caccioppoli and Scorza Dragoni [1] introduced the concept of regular approximate differential (Definition 3, below) and proved that in order for a continuous real-valued function defined on a planar open set to have a regular approximate differential a.e., it suffices that the function have first partial derivatives a.e. In this note we extend this result in a natural way to n-space by proving that, again for a continuous function on an open set, a sufficient condition for the existence a.e. of a regular approximate differential is the existence a.e. of a total differential with respect to all but one variable. This strengthens a result of Lodigiani [2], who required that the function have partial derivatives everywhere which are continuous with respect to all but one variable.

2. Notation, definitions and basic theorems. Let \( p = (p_1, \ldots, p^n) \) denote points in n-space \( R^n \), and \( \|p - q\| = (\sum_{j=1}^{n} (p^j - q^j)^2)^{1/2} \) the distance between \( p \) and \( q \). We use \( \nabla f(p) \) for the n-tuple of partial derivatives \( (f'_1(p), \ldots, f'_n(p)) \) and set

\[
\nabla f(p) \cdot (p - q) = \sum_{j=1}^{n} f'_j(p)(p^j - q^j).
\]

The notation \( mS \) will denote n-dimensional Lebesgue measure of a set \( S \) in \( R^n \).

Definition 1. Given a real-valued function \( f : S \to R \) on a subset \( S \) of \( R^n \), suppose \( f \) has partial derivatives at a point \( p \) in \( S \). Let

\[
e(p, q) = \frac{f(q) - f(p) - \nabla f(p) \cdot (q - p)}{\|q - p\|}, \quad p \ne q.
\]
Then $f$ is said to have a total differential (in the Stolz sense) at $p$ if 
$\lim_{q \to p} e(p,q) = 0$.

**Definition 2.** Given any point $p$ in $\mathbb{R}^n$ let $\mathcal{C}(p)$ denote any family of oriented $n$-cubes (edges parallel to the coordinate axes) such that (i) $p$ is the center of each $n$-cube of $\mathcal{C}(p)$, and (ii) $p$ is a point of density for $\bigcup \partial C$, $C \in \mathcal{C}(p)$, where $\partial C$ denotes the frontier of $C$. We term $\mathcal{C}(p)$ a thick regular family of $n$-cubes at $p$.

The following lemma is easily verified.

**Lemma 1.** If $p$ is a point of (linear) density of a subset $S$ of $\mathbb{R}^n$ in the direction of every coordinate axis, then there exists a thick regular family of $n$-cubes $\mathcal{C}(p)$ such that the lines through $p$ parallel to the coordinate axes intersect the faces of each $n$-cube in a point of $S$.

**Definition 3.** Given a real-valued function $f: S \to \mathbb{R}$, where $S$ is a subset of $\mathbb{R}^n$, we say that $f$ has a regular approximate differential at a point $p$ in $S$ if there exists a thick regular family $\mathcal{C}(p)$ such that $f$ restricted to $\bigcup (S \cap \partial C)$, $C \in \mathcal{C}(p)$, has a total differential at $p$.

**Lemma 2.** Let $p$ be a point of density of a measurable set $S$ in $\mathbb{R}^n$. Then for every $\eta > 0$ there exists a $\delta > 0$ such that whenever $\|p - q\| < \delta$ for a point $q$ in $\mathbb{R}^n$ there corresponds a point $q^*$ in $S$ satisfying the inequality $\|q - q^*\| < \eta \|p - q\|$.

**Proof.** This is a simple exercise in the definition of density, and may be found implicitly in Rademacher [3].

**Corollary.** Let $S$ be a bounded measurable set in $\mathbb{R}^n$, and let $\eta > 0$ be given. Then for every $\epsilon > 0$ there corresponds a closed subset $F$ of $S$ and a number $\delta > 0$ for which the following two properties hold:

(i) $m(S - F) < \epsilon$.

(ii) For every $p$ in $F$, $q$ in $\mathbb{R}^n$ with $\|p - q\| < \delta$ and $p^k = q^k$ for some integer $k$, there corresponds a point $q^*_*$ in $S$ such that $q^*_* = q^k$ and $\|q^*_* - q\| < \eta \|p - q\|$.

**Proof.** This follows easily from Lemma 2 by applying it to $(n-1)$-dimensional sections of $S$ and choosing the set $F$ to uniformize the $\delta$.

**Lemma 3.** Let $f: S \to \mathbb{R}$ be a continuous real-valued function defined on a bounded open set $S$ with the following property: for almost every point $p \in S$ there exist $\delta > 0$, $M > 0$ such that

(*) $|f(p) - f(q)| \leq M \|p - q\|$ whenever $q \in S$, $\|p - q\| < \delta$ and $p^k = q^k$ for some $k$.

Then for every $\epsilon > 0$ there exists a measurable subset $E$ of $S$, and numbers $\delta > 0$, $M > 0$ such that (1) $m(S - E) < \epsilon$, (2) for every $p \in E$ condition (*) in the hypothesis holds.
Proof. Define the set $E_n$ to consist of all points $p$ in $S$ such that $|f(p) - f(q)| \leq m\|p - q\|$ if $q \in S$, $\|p - q\| < 1/m$ and $p_k = q_k$ for some $k$. Clearly, $E_n$ is an ascending sequence of sets with $m(S - \bigcup E_n) = 0$ in view of the hypothesis. To show that each $E_n$ is closed relative to $S$ let $p_n$ denote any sequence of points of $E_n$ converging to $p_0 \in S$. Suppose $q \in S$, $q_k = p_k$, and $\|p_0 - q\| < 1/m$. Then, for $n$ large enough, $\|p_n - q\| < 1/n$. Let $\bar{p}_n$ denote the point such that $\bar{p}_n = q_k = p_k$, but $\bar{p}_n = p_n$ for $j \neq k$. Then, for $n$ large enough, $\bar{p}_n \in S$ since $S$ is open, and so $|f(\bar{p}_n) - f(q)| \leq m\|\bar{p}_n - q\|$. Since $f$ is continuous on $S$ and clearly $p_n \to p_0$ we infer that

$$|f(p_0) - f(q)| = m\|p_0 - q\|$$

and so $p_0 \in E_n$, and the proof that $E_n$ is closed relative to $S$ is complete. Thus for any assigned $\varepsilon > 0$ there is an $m_0$ such that $m(S - E_m) < \varepsilon$. We therefore take $E = E_m$, $\delta = 1/m_0$ and $M = m_0$, and the proof is complete.

3. Main theorem.

Theorem. Let $f : S \to \mathbb{R}$ be a continuous real-valued function defined on an open bounded set $S$ in $\mathbb{R}^n$. Assume that $f$ has a total differential a.e. in the direction of every $(n-1)$-dimensional coordinate hyperplane. Then $f$ has a regular approximate differential a.e. on $S$.

Proof. Let $\varepsilon > 0$, $\gamma > 0$ be given arbitrarily. In view of the hypotheses of the theorem, the conditions of Lemma 3 are satisfied. Select a set $E = E_1$ and numbers $\delta = \delta_1$ and $M = M_1$, satisfying the conclusion of Lemma 3. By hypothesis the partial derivatives exist a.e. on $S$ and hence by the well-known theorem of Stepanoff [7], there exists a measurable subset $E_2$ of $E_1$ such that $f$ has a total differential at each point of $E_2$ with respect to $E_2$ and $m(E_1 - E_2) < \varepsilon$. By Lusin's theorem we may assume without loss that $E_2$ is closed and the partial derivatives are continuous on $E_2$, hence $|\nabla f(p)| \leq M_2$ for some constant $M_2$ and every $p \in E_2$. Further, let $E_3$ denote the set of all points of $E_2$ which are points of density of $E_1$ in the direction of every coordinate axis and in the direction of every $(n-1)$-dimensional coordinate hyperplane. Then $m(E_2 - E_3) = 0$ (see Saks [6, p. 298] for the linear case). Finally, let $E_4$ be (see corollary of Lemma 2) a measurable subset of $E_3$ and let $\delta_2 > 0$ chosen so that

(i) $m(E_3 - E_4) < \varepsilon$,

(ii) for every $p \in E_4$ and $q \in S$ with $\|p - q\| < \delta_2$ and $p_k = q_k$ for some $k$, there corresponds a $q_* \in E_1$ such that

$$\|q - q_*\| < \frac{\gamma}{3M} \|p - q\| \& q_*^k = q^k$$

where $M = \max(M_1, M_2)$.  

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Consider now any point $p_0 \in E_4$, and let $\delta_3 > 0$ be chosen so that
\begin{equation}
|f(\tilde{q}) - f(p_0) - \nabla f(p_0) \cdot (\tilde{q} - p_0)| \leq (\gamma/6) \|\tilde{q} - p_0\|
\end{equation}
when $\|\tilde{q} - p_0\| < \delta_3$, $\tilde{q} \in E_2$.

Take now any point $p_0 \in E_4$. Let $\mathcal{C}(p_0)$ denote a thick regular family of $n$-cubes centered at $p_0$ such that (see Lemma 1) the lines through $p_0$ parallel to the coordinate axes intersect the frontiers of the $n$-cubes in points of $E_3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and take any point $q \in S$ on the frontier of any $n$-cube $C$ of $\mathcal{C}(p_0)$ having diagonal-length less than $\delta$. Suppose without loss of generality that the 1-axis is perpendicular to the face whereon $q$ lies. Then $q = (p_0^1 + h, q_2, q_3, \ldots, q_n)$ for some number $h$. Let $p_4 = (p_0^1 + h, p_0^2, \ldots, p_0^n)$. Then $p_4 \in E_3$, so that since $\|q - p_4\| < \delta_1$ we have a $q* = (p_4^1 + h, q_2^*, \ldots, q_n^*) \in E_3$ satisfying (1). Then $\|q - q*\| < \delta_1$ and $|q* - p_0| < \delta_3$, and so we have the following sequences of inequalities:
\begin{align*}
|f(q) - f(p_0) - \nabla f(p_0) \cdot (q - p_0)| &\leq |f(q) - f(q*)| + |f(q*) - f(p_0) - \nabla f(p_0) \cdot (q - p_0)| \\
&\leq M_1 \|q - q*\| + |f(q*) - f(p_0) - \nabla f(p_0) \cdot (q* - p_0)| \\
&\quad + |\nabla f(p)| \cdot \|q - q*\| \\
&\leq M_4 (\gamma/3M_1) \|q - p_1\| + (\gamma/6) \|q* - p_0\| + M_2 (\gamma/3M_2) \|q - p_1\| \\
&\leq (\gamma/3) \|q - p_0\| + (\gamma/3) \|q - p_0\| + (\gamma/3) \|q - p_0\| = \gamma \|q - p_0\|.
\end{align*}

Thus, $f$ has a total differential at $p_0$ relative to the part of $S$ in $\bigcup \partial C$, $C \in \mathcal{C}(p_0)$, and so $f$ has a regular approximate differential at $p_0$. Since $p_0$ is an arbitrary point of $E_4$ and $m(S - E_4) < 3\epsilon$, the theorem is proved.

REFERENCES


DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, AMHERST, NEW YORK 14226