ALL $\alpha$-CONVEX FUNCTIONS ARE UNIVALENT AND STARLIKE

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ABSTRACT. The authors show that $\alpha$-convex functions are starlike, $-\infty \leq \alpha \leq \infty$, thus extending some earlier results.

In recent notes we all proved that $\alpha$-convex functions are univalent and starlike for $0 \leq \alpha \leq \infty$ ([2], [3], [4]). In this note we offer another proof, one that is valid for all $\alpha$, $-\infty \leq \alpha \leq \infty$.

Let $f(z) = z + a_2 z^2 + \cdots$ be analytic in the unit disc $\Delta$, with $f(z)f'(z)/z \neq 0$ there, and let $\alpha$ be a real number. Then $f(z)$ is said to be $\alpha$-convex in $\Delta$ if and only if the inequality

$$\text{Re} \left[ (1 - \alpha)z \frac{f'(z)}{f(z)} + \alpha \left( 1 + z \frac{f''(z)}{f'(z)} \right) \right] > 0$$

holds in $\Delta$ [1]. For such functions we obtain the following result.

**Theorem.** If $f(z) = z + \cdots$ is $\alpha$-convex in the unit disc $\Delta$, then $f(z)$ is starlike and univalent in $\Delta$. Moreover, if $\alpha \geq 1$, then $f(z)$ is convex for $|z| < 1$, and if $\alpha \leq -1$, then $1/f(1/z)$ is convex for $|z| > 1$.

**Proof.** If we set $p(z) = zf'(z)/f(z)$ in (1), then we obtain

$$\text{Re} \left[ p(z) - i\alpha(\partial/\partial \theta) \ln p(z) \right] > 0, \quad z = re^{i\theta},$$

which holds for all $z$ in $\Delta$. Suppose that there exists a point $z_0 = r_0 e^{i\theta_0}$ in $\Delta$ such that $\text{Re} p(z) \geq 0$ for $|z| \leq r_0$ and $\text{Re} p(z_0) = 0$. Then $\arg p(r_0 e^{i\theta})$ has either a maximum or a minimum for $\theta = \theta_0$. Hence $(\partial/\partial \theta) \arg p(z_0) = 0$. If we combine this last remark with $\text{Re} p(z_0) = 0$, then we conclude that the left-hand member of (2), and hence of (1), must vanish for $z = z_0$. But this is a contradiction of (1) (and of (2)). Since $p(0) = 1$ and $\text{Re} p(z)$ does not vanish in $\Delta$, we conclude that $p(z) = z f'(z)/f(z)$ has a positive real part in $\Delta$. Hence $f(z)$ is univalent and starlike in $\Delta$ for all $\alpha$, $-\infty < \alpha < \infty$.

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Now set \( z = 1/\zeta \), \( g(\zeta) = 1/f(1/\zeta) \) in (1). We obtain the inequality

\[
\Re \left[ (1 + \alpha) \frac{g'(\zeta)}{g(\zeta)} - \alpha \left( 1 + \zeta \frac{g''(\zeta)}{g'(\zeta)} \right) \right] > 0,
\]

which must hold for all \(|\zeta| > 1\). Since \( f(z) \) is univalent and starlike with respect to the origin, so is \( g(\zeta) \). Hence we obtain the inequality \( \Re \{1 + \zeta [g''(\zeta)/g'(\zeta)]\} > 0 \), provided \( 1 + \alpha \leq 0 \), from (3). Hence \( g(\zeta) = 1/f(1/\zeta) \) is convex for \(|\zeta| > 1\), \( \alpha \leq -1 \).

If \( \alpha \geq 1 \), then the first term on the left-hand side of (1) is nonpositive. From this it follows that \( f(z) \) is convex.

If \( \alpha = \pm \infty \), then (1), with application of the maximum principle for harmonic functions, implies that \( f(z) \equiv z \).

The proof is now complete.

REMARK. It is easy to see that if \( f(z) \) is \( \alpha_0 \)-convex, then \( f(z) \) is \( \alpha \)-convex for (i) \( 0 \leq \alpha \leq \alpha_0 \), if \( 0 \leq \alpha_0 \), (ii) \( \alpha_0 \leq \alpha \leq 0 \), if \( \alpha_0 \leq 0 \).

REFERENCES


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