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ALL α-CONVEX FUNCTIONS ARE UNIVALENT AND STARLIKE

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ABSTRACT. The authors show that α-convex functions are starlike, $-\infty \leqq \alpha \leqq \infty$, thus extending some earlier results.

In recent notes we all proved that α-convex functions are univalent and starlike for $0 \leqq \alpha \leqq \infty$ ([2], [3], [4]). In this note we offer another proof, one that is valid for all α, $-\infty \leqq \alpha \leqq \infty$.

Let $f(z)=z+a_2z^2+\cdots$ be analytic in the unit disc Δ, with $f(z)f'(z)/z \neq 0$ there, and let α be a real number. Then $f(z)$ is said to be α-convex in Δ if and only if the inequality

$$\operatorname{Re}\left[(1-\alpha)z\frac{f'(z)}{f(z)}+\alpha\left(1+z\frac{f''(z)}{f'(z)}\right)\right]>0 \tag{1}$$

holds in Δ [1]. For such functions we obtain the following result.

THEOREM. *If $f(z)=z+\cdots$ is α-convex in the unit disc Δ, then $f(z)$ is starlike and univalent in Δ. Moreover, if $\alpha \geqq 1$, then $f(z)$ is convex for $|z|<1$, and if $\alpha \leqq -1$, then $1/f(1/z)$ is convex for $|z|>1$.*

PROOF. If we set $p(z)=zf'(z)/f(z)$ in (1), then we obtain

$$\operatorname{Re}[p(z)-i\alpha(\partial/\partial\theta)\ln p(z)]>0, \qquad z=re^{i\theta}, \tag{2}$$

which holds for all z in Δ. Suppose that there exists a point $z_0=r_0e^{i\theta_0}$ in Δ such that $\operatorname{Re} p(z) \geqq 0$ for $|z| \leqq r_0$ and $\operatorname{Re} p(z_0)=0$. Then $\arg p(r_0e^{i\theta})$ has either a maximum or a minimum for $\theta=\theta_0$. Hence $(\partial/\partial\theta)\arg p(z_0)=0$. If we combine this last remark with $\operatorname{Re} p(z_0)=0$, then we conclude that the left-hand member of (2), and hence of (1), must vanish for $z=z_0$. But this is a contradiction of (1) (and of (2)). Since $p(0)=1$ and $\operatorname{Re} p(z)$ does not vanish in Δ, we conclude that $p(z)=zf'(z)/f(z)$ has a positive real part in Δ. Hence $f(z)$ is univalent and starlike in Δ for all α, $-\infty<\alpha<\infty$.

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Now set $z=1/\zeta$, $g(\zeta)=1/f(1/\zeta)$ in (1). We obtain the inequality

$$\operatorname{Re}\left[(1+\alpha)\zeta\frac{g'(\zeta)}{g(\zeta)}-\alpha\left(1+\zeta\frac{g''(\zeta)}{g'(\zeta)}\right)\right]>0, \tag{3}$$

which must hold for all $|\zeta|>1$. Since $f(z)$ is univalent and starlike with respect to the origin, so is $g(\zeta)$. Hence we obtain the inequality $\operatorname{Re}\{1+\zeta[g''(\zeta)/g'(\zeta)]\}>0$, provided $1+\alpha\leqq 0$, from (3). Hence $g(\zeta)=1/f(1/\zeta)$ is convex for $|\zeta|>1$, $\alpha\leqq -1$.

If $\alpha\geqq 1$, then the first term on the left-hand side of (1) is nonpositive. From this it follows that $f(z)$ is convex.

If $\alpha=\pm\infty$, then (1), with application of the maximum principle for harmonic functions, implies that $f(z)\equiv z$.

The proof is now complete.

REMARK. It is easy to see that if $f(z)$ is α_0-convex, then $f(z)$ is α-convex for (i) $0\leqq\alpha\leqq\alpha_0$, if $0\leqq\alpha_0$, (ii) $\alpha_0\leqq\alpha\leqq 0$, if $\alpha_0\leqq 0$.

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