SOME ALGEBRAIC K-THEORETIC APPLICATIONS OF THE LF AND NF FUNCTORS

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Abstract. Using previous results of H. Bass, we compute $\text{Pic}(P^1(R))$ and $L^nN^0K_0(\Lambda[G])$ where $R$ is a commutative ring, $\Lambda$ any commutative finite algebra over a Dedekind ring, and $G$ any finitely generated free abelian group or monoid.

Introduction. This paper is a sequel to some results on LF and NF functors introduced by Bass in [1]. The notations are those of [1]. In §1, some results are given on Picard group of the projective line and §2 deals with $L^nN^0K_0(\Lambda)$ and $L^nN^0K_0(\Lambda[G])$ for $\Lambda$ any commutative finite algebra over a Dedekind domain and $G$ a free abelian group or monoid. Lastly we observe that $L^nN^0K_0(A)$ is a filtered $K_0(R)$-module if $A$ is an algebra over a commutative ring $R$.

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1. Let $R$ be a commutative ring, $\text{Pic}(R)$ the category (with product $\otimes$) of finitely generated projective modules of rank 1 and $(T, T_\pm)$ an oriented cycle. We write $\text{Pic}(P^1(R))$ for the fibre product category

$$\text{Pic}(R[T_+]) \times_{\text{Pic}(R)} \text{Pic}(R[T_-])$$

and denote $K_0(\text{Pic}(P^1(R)))$ by $\text{Pic}(P^1(R))$.

Theorem 1.1. For $P \in \text{Pic}(R[T_+])$, $P_0 = P \otimes_{R[T_+]} R \in \text{Pic}(R)$; let $P \approx P_0[T_+]$. Then $\text{Pic}(P^1(R)) \approx H_0(R) \oplus \text{Pic}(R)$.

Proof. From [1, p. 365, Theorem 4.3], we obtain an exact sequence

$$K_1(\text{Pic}(P^1(R))) \to U(R[T_+]) \oplus U(R[T_-]) \to U(R[T])$$

$$(I) \to \text{Pic}(P^1(R)) \to \text{Pic}(R[T_+]) \oplus \text{Pic}(R[T_-]) \to \text{Pic}(R[T])$$

since $K_1(\text{Pic}(R)) \approx U(R)$ and $K_0(\text{Pic}(R)) \approx \text{Pic}(R)$.
Also by [1, p. 670, Corollary 7.7] and the fact that \( LU \approx H_0 \) (see [1, p. 671, Proposition 7.8]) we obtain from (I) the following exact sequence

\[
\text{(II) } 0 \rightarrow H_0(R) \rightarrow \text{Pic}(P^1(R)) \rightarrow \text{Pic}(R) \rightarrow 0.
\]

Now define \( \eta: \text{Pic}(R) \rightarrow \text{Pic}(P^1(R)) \) by \( \eta[P] = (P[T_+], 1_{F[T]}, P[T_-]) \). So \( j \eta[P] = j(P[T_+], 1_{F[T]}, P[T_-]) = [P[T_+] \otimes_{R[T]} R] = [P] \). So the sequence (II) is split exact and hence \( \text{Pic}(P^1(R)) \approx \text{Pic}(R) \oplus H_0(R) \).

**Corollary 1.2.** If \( R \) is a commutative-Noetherian ring of stable Serre dimension \( \leq 1 \) then \( \text{Pic}(P^1(R)) \approx K_0(R) \).

**Proof.** Follows from \( K_0(R) \approx H_0(R) \oplus \text{rk}_0(R) \) and the fact that \( \text{rk}_0(R) \approx \text{Pic}(R) \) if and only if stable Serre dimension \( R \leq 1 \) ([2, p. 59]).

**Corollary 1.3.** Suppose \( R \) is a commutative Artinian ring. Then \( \text{Pic}(P^1(R)) \approx K_0(P^1(R)) \approx H_0(R) \). When the cartesian square

\[
P(P^1(R)) \rightarrow P(R[T_-])
\]

\[
\downarrow \hspace{1cm} \downarrow
\]

\[
P(R[T_+]) \rightarrow P(R[T])
\]

is \( E \)-surjective, then \( \text{Pic}(P^1(R)) \approx K_0(P^1(R)) \).

**Proof.** Since \( \tau_{\pm}: R[T_\pm] \rightarrow R[T] \) are inclusions, \( H_0(R[T_\pm]) \rightarrow H_0(R[T]) \) are injective and we can replace the \( K_0 \)'s in the exact sequence for \( K_0(P^1(R)) \) in [1, p. 679], by \( \text{rk}_0 \)'s (see [1, p. 466]). The resulting exact sequence is then mapped into (I) in the proof of 1.1 and the result follows by applying 1.1 and Lemma 7.6 of [2].

2. Let \( R \) be the category of rings (with unit) and \( \text{Ab} \) the category of Abelian groups.

**Lemma 2.1.** If a functor \( F: R \rightarrow \text{Ab} \) has the property that \( F(R) \rightarrow F(R/N) \) is an isomorphism when \( N \) is a nilpotent ideal of \( R \), then \( LF, NF \) have the same property. Hence \( L^\infty N^\infty K_0 \) has this property.

**Proof** is easy and is omitted.

**Lemma 2.2 ([1, p. 163]).** Let \( R \) be a Dedekind ring with quotient field \( L \). Suppose \( \Lambda \) is a finite \( R \)-algebra. Then there is a largest two-sided nilpotent ideal \( N \) in \( \Lambda \). If \( \Gamma \) is the \( R \)-torsion submodule of \( \Lambda/N \), then \( \Gamma \) is a semisimple ring and \( \Lambda/N \approx \Gamma \times A \) where \( A \) is an \( R \)-order in a semisimple algebra.

**Theorem 2.3.** Let \( R \) be a Dedekind ring with quotient field \( L \), \( \Lambda \) any commutative finite \( R \)-algebra, \( \Gamma, A \) as in 2.2, \( G \) a finitely generated free
abelian group or monoid. Then

(i) \( L^nNiK_0(\Lambda) \approx L^nNiK_0(\Lambda[G]) = 0 \) for \( n > 0 \) and \( i > 0 \), or for \( n > 1 \) and \( i \geq 0 \),

(ii) \( \det_0(\Lambda[G]) : RK_0(\Lambda[G]) \rightarrow \text{Pic}(\Lambda[G]) \) is an isomorphism,

(iii) \( LK_0(\Lambda) \approx LK_0(\Lambda[G]) \approx LK_0(\Lambda) \) is a torsion free abelian group,

(iv) \( K_0(\Lambda[G]) \approx H_0(\Gamma) \oplus K_0(A[G]) \),

(v) \( N^iK_0(\Lambda) \approx N^iK_0(\Lambda) \).

Similarly \( N^iK_0(\Lambda[G]) \approx N^iK_0(\Lambda[G]) \).

**Proof.** By 2.1 we have \( L^nNiK_0(\Lambda) \approx L^nNiK_0(\Lambda/N) \). Also

\[
L^nNiK_0(\Lambda[G]) \approx L^nNiK_0(\Lambda/N[G])
\]

from (2.1) and Grothendieck's theorem [1, p. 636]. Since \( \Lambda/N = \Gamma \times A \),

the above theorem reduces to \( \Lambda = \Gamma \) and \( A = A \).

So (i) follows from [1, p. 688, Theorem 10.2]; (ii) follows from [1, p. 690, Theorem 10.4]; (iii) follows from [1, p. 690, 10.4(c)], Grothen- dicks' theorem, and [2, Lemma 7.6]; (iv) follows from Grothendieck's theorem, and \( Rk_0(\Gamma) = \text{Pic}(\Gamma) = 0 \), \( \Gamma \) being semisimple; (v) follows from [1, p. 685, Theorem 10.1].

**Corollary 2.4.** Suppose \( R, \Lambda, \Gamma \) are as in 2.3, then \( \text{Pic}(P^i(\Lambda)) \approx K_0(\Lambda) \). If \( R = \mathbb{Z} \) or \( F[t] \), the polynomial ring in \( t \) over a finite field, then \( \text{Pic}(P^i(\Lambda)) \) is a finitely generated abelian group.

**Proof.** Follows from the union of 1.1, 2.3 and [1, p. 545, Theorem 2.7].

3. Let \( A \) be an algebra over a commutative ring \( R \). In [1, p. 473], Bass defined a \( K_0(R) \)-module filtration \( F^iRk_0(A) \) on \( K_0(A) \) using the space \( \text{max}(R) \) of maximal ideals of \( R \).

We now observe the following:

3.1. If a functor \( F \) on \( R \)-algebras has a natural filtration, so do \( LF \) and \( NF \). So, if \( F \) is a filtered \( K_0(R) \)-module so are \( LF \) and \( NF \). Hence \( L^nNiK_0(A) \) is a filtered \( K_0(R) \)-module.

Proof is easy and is omitted.

**References**


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