SOME ALGEBRAIC $K$-THEORETIC APPLICATIONS OF THE LF AND NF FUNCTORS

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Abstract. Using previous results of H. Bass, we compute $\text{Pic}(P'(R))$ and $L^nN'K_0(\Lambda[G])$ where $R$ is a commutative ring, $\Lambda$ any commutative finite algebra over a Dedekind ring, and $G$ any finitely generated free abelian group or monoid.

Introduction. This paper is a sequel to some results on LF and NF functors introduced by Bass in [1]. The notations are those of [1]. In §1, some results are given on Picard group of the projective line and §2 deals with $L^nN'K_0(\Lambda)$ and $L^nN'K_0(\Lambda[G])$ for $\Lambda$ any commutative finite algebra over a Dedekind domain and $G$ a free abelian group or monoid. Lastly we observe that $L^nN'K_0(A)$ is a filtered $K_0(R)$-module if $A$ is an algebra over a commutative ring $R$.

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1. Let $R$ be a commutative ring, $\text{Pic}(R)$ the category (with product $\otimes$) of finitely generated projective modules of rank 1 and $(T, T_{\pm})$ an oriented cycle. We write $\text{Pic}(P(R))$ for the fibre product category

$$\text{Pic}(R[T_+]) \times_{\text{Pic}(R[T])} \text{Pic}(R[T_-])$$

and denote $K_0(\text{Pic}(P(R)))$ by $\text{Pic}(P(R))$.

Theorem 1.1. For $P \in \text{Pic}(R[T_+])$, $P_0=P \otimes_R T_1 \in \text{Pic}(R)$; let $P \approx P_0[T_+]$. Then $\text{Pic}(P_1(R)) \approx H_0(R) \oplus \text{Pic}(R)$.

Proof. From [1, p. 365, Theorem 4.3], we obtain an exact sequence

$$K_1(\text{Pic}(P(R))) \rightarrow U(R[T_+]) \oplus U(R[T_-]) \rightarrow U(R[T])$$

$$(I) \rightarrow \text{Pic}(P(R)) \rightarrow \text{Pic}(R[T_+]) \oplus \text{Pic}(R[T_-]) \rightarrow \text{Pic}(R[T]),$$

since $K_1(\text{Pic}(R)) \approx U(R)$ and $K_0(\text{Pic}(R)) \approx \text{Pic}(R)$. 

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Also by [1, p. 670, Corollary 7.7] and the fact that $LU \approx H_0$ (see [1, p. 671, Proposition 7.8]) we obtain from (I) the following exact sequence

$$0 \rightarrow H_0(R) \rightarrow \text{Pic}(P_1(R)) \rightarrow \text{Pic}(R) \rightarrow 0.$$  

(II)

Now define $\eta: \text{Pic}(R) \rightarrow \text{Pic}(P_1(R))$ by $\eta[P] = (P[T_+], 1_{P(T_+)}, P[T_-])$. So $\eta[P] = (P[T_+], 1_{P(T_+)}), P(T_-) = [P[T_+] \otimes_{R[T_+]} R] = [P]$. So the sequence (II) is split exact and hence $\text{Pic}(P_1(R)) \approx \text{Pic}(R) \oplus H_0(R)$.

**Corollary 1.2.** If $R$ is a commutative-Noetherian ring of stable Serre dimension $\leq 1$ then $\text{Pic}(P_1(R)) \approx K_0(R)$.

**Proof.** Follows from $K_0(R) \approx H_0(R) \oplus Rk_0(R)$ and the fact that $Rk_0(R) \approx \text{Pic}(R)$ if and only if stable Serre dimension $R \leq 1$ ([2, p. 59]).

**Corollary 1.3.** Suppose $R$ is a commutative Artinian ring. Then $\text{Pic}(P_1(R)) \approx K_0(P_1(R)) \approx H_0(R)$. When the cartesian square

$$
\begin{array}{ccc}
P(P_1(R)) & \rightarrow & P(R[T_-]) \\
\downarrow & & \downarrow \\
P(R[T_+]) & \rightarrow & P(R[T])
\end{array}
$$

is E-surjective, then $\text{Pic}(P_1(R)) \approx K_0(P_1(R))$.

**Proof.** Since $\tau_{\pm}: R[T_{\pm}] \rightarrow R[T]$ are inclusions, $H_0(R[T_{\pm}]) \rightarrow H_0(R[T])$ are injective and we can replace the $K_0$'s in the exact sequence for $K_0(P_1(R))$ in [1, p. 679], by $Rk_0$'s (see [1, p. 466]). The resulting exact sequence is then mapped into (I) in the proof of 1.1 and the result follows by applying 1.1 and Lemma 7.6 of [2].

2. Let $R$ be the category of rings (with unit) and $\text{Ab}$ the category of Abelian groups.

**Lemma 2.1.** If a functor $F: R \rightarrow \text{Ab}$ has the property that $F(R) \rightarrow F(R/N)$ is an isomorphism when $N$ is a nilpotent ideal of $R$, then $LF, NF$ have the same property. Hence $L^nN^kK_0$ has this property.

**Proof** is easy and is omitted.

**Lemma 2.2** ([1, p. 163]). Let $R$ be a Dedekind ring with quotient field $L$. Suppose $\Lambda$ is a finite $R$-algebra. Then there is a largest two-sided nilpotent ideal $N$ in $\Lambda$. If $\Gamma$ is the $R$-torsion submodule of $\Lambda/N$, then $\Gamma$ is a semisimple ring and $\Lambda/N \approx \Gamma \times A$ where $A$ is an $R$-order in a semisimple algebra.

**Theorem 2.3.** Let $R$ be a Dedekind ring with quotient field $L$, $\Lambda$ any commutative finite $R$-algebra, $\Gamma, A$ as in 2.2, $G$ a finitely generated free
abelian group or monoid. Then

(i) \( L^n\mathcal{N}\mathcal{K}_0(\Lambda) \cong L^n\mathcal{N}\mathcal{K}_0(\Lambda[G]) = 0 \) for \( n > 0 \) and \( i > 0 \), or for \( n > 1 \) and \( i \geq 0 \),

(ii) \( \det_\theta(\Lambda[G]) : R\mathcal{K}_0(\Lambda[G]) \to \text{Pic}(\Lambda[G]) \) is an isomorphism,

(iii) \( L\mathcal{K}_0(\Lambda) \cong L\mathcal{K}_0(\Lambda[G]) \cong \mathcal{K}_0(\Lambda[G]) \) is a torsion free abelian group,

(iv) \( \mathcal{K}_0(\Lambda[G]) \cong H_0(\Gamma) \oplus \mathcal{K}_0(\Lambda[G]) \),

(v) \( N^i\mathcal{K}_0(\Lambda) \approx N^i\mathcal{K}_0(\Lambda[G]) \).

Similarly \( N^i\mathcal{K}_0(\Lambda[G]) \approx N^i\mathcal{K}_0(\Lambda[G]) \).

**Proof.** By 2.1 we have \( L^n\mathcal{N}\mathcal{K}_0(\Lambda) \cong L^n\mathcal{N}\mathcal{K}_0(\Lambda/N) \). Also

\[
L^n\mathcal{N}\mathcal{K}_0(\Lambda[G]) \approx L^n\mathcal{N}\mathcal{K}_0(\Lambda/N[G])
\]

from (2.1) and Grothendieck's theorem [1, p. 636]. Since \( \Lambda/N = \Gamma \times A \), the above theorem reduces to \( \Lambda = \Gamma \) and \( \Lambda = A \).

So (i) follows from [1, p. 688, Theorem 10.2]; (ii) follows from [1, p. 690, Theorem 10.4]; (iii) follows from [1, p. 690, 10.4(c)], Grothendieck's theorem, and [2, Lemma 7.6]; (iv) follows from Grothendieck's theorem, and \( R\mathcal{K}_0(\Gamma) = \text{Pic}(\Gamma) = 0 \), \( \Gamma \) being semisimple; (v) follows from [1, p. 685, Theorem 10.1].

**Corollary 2.4.** Suppose \( R, \Lambda, \Gamma \) are as in 2.3, then \( \text{Pic}(P^1(\Lambda)) \approx \mathcal{K}_0(\Lambda) \). If \( R = \mathbb{Z} \) or \( F[[t]] \), the polynomial ring in \( t \) over a finite field, then \( \text{Pic}(P^1(\Lambda)) \) is a finitely generated abelian group.

**Proof.** Follows from the union of 1.1, 2.3 and [1, p. 545, Theorem 2.7].

3. Let \( A \) be an algebra over a commutative ring \( R \). In [1, p. 473], Bass defined a \( \mathcal{K}_0(\mathbb{R}) \)-module filtration \( F_\mathbb{R}\mathcal{K}_0(A) \) on \( \mathcal{K}_0(A) \) using the space \( \text{max}(R) \) of maximal ideals of \( R \).

We now observe the following:

3.1. If a functor \( F \) on \( R \)-algebras has a natural filtration, so do \( LF \) and \( NF \). So, if \( F \) is a filtered \( \mathcal{K}_0(R) \)-module so are \( LF \) and \( NF \). Hence \( L^n\mathcal{N}\mathcal{K}_0(\Lambda) \) is a filtered \( \mathcal{K}_0(\mathbb{R}) \)-module.

Proof is easy and is omitted.

**References**


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