A PROPERTY OF PROJECTIVE IDEALS
IN SEMIGROUP ALGEBRAS

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Abstract. The condition that certain left ideals in a finite monoid generate projective left ideals in the semigroup algebra imposes a strong restriction on the intersection of principal left ideals in the semigroup.

Let $S$ be a finite monoid, $k$ be a commutative ring with identity, and let $I \subseteq S$ be a left ideal in $S$. We demand that $kI$ be projective as a left $kS$-module and investigate the resulting restrictions on the structure of $I$. In particular we can look for necessary conditions on $S$ for $kS$ to be left hereditary. (A sufficient condition is obtained in [4] and [5].) Semigroup terminology below follows [1] and [2].

We first need the following facts, which are valid in any ring with identity.

Lemma 1. Let $R$ be a ring with identity. Let $I \subseteq R$ be a left ideal which is projective as a left $R$-module. Let $e \in R$ be any idempotent. Then the left ideal $I + Re$ is projective if and only if $I \cap Re$ is a direct summand of $I$.

Proof. We observe that we have the following two short exact sequences:

$$0 \rightarrow I(1 - e) \rightarrow I + Re \rightarrow Re \rightarrow 0,$$

$$0 \rightarrow I \cap Re \rightarrow I \rightarrow I(1 - e) \rightarrow 0,$$

where the map on the right end of the first sequence is $x \mapsto xe$, which has kernel $(I + Re) \cap R(1 - e) = I(1 - e)$, and the map on the right end of the second sequence is $x \mapsto x(1 - e)$. Since $Re$ is projective, the first sequence always splits, so that $I + Re$ is projective if and only if $I(1 - e)$ is. On the other hand, since $I$ is projective, $I(1 - e)$ is projective if and only if the second sequence splits.

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Corollary. If \( R \) is a ring with identity and if \( e_1 \) and \( e_2 \) are idempotents of \( R \), then \( Re_1 + Re_2 \) is a projective left ideal of \( R \) if and only if \( Re_1 \cap Re_2 \) has an idempotent generator.

Proof. By the lemma, \( Re_1 \cap Re_2 \) is a direct summand of \( Re_1 \). If \( \pi : Re_1 \to Re_1 \cap Re_2 \) is a retraction of \( Re_1 \) onto \( Re_1 \cap Re_2 \), then \( f = \pi(e) \) is the desired idempotent since \( f^2 = f\pi(e) = \pi(f) = f \) and \( Rf = \pi(Re_1) = Re_1 \cap Re_2 \).

Now let \( S \) be a finite monoid, \( e_1, e_2 \in S \) be idempotents, and \( k \) be a commutative ring with identity. Then we have the following:

Theorem 1. (i) If the left ideal \( kSe_1 + kSe_2 \subseteq kS \) is projective as a left \( kS \)-module, then the semigroup left ideal \( Se_1 \cap Se_2 \subseteq S \) is generated by idempotents of \( S \), or \( Se_1 \cap Se_2 = \emptyset \).

(ii) Furthermore, if \( L \) is an \( \mathcal{L} \)-class of \( S \) maximal in the \( \mathcal{L} \)-class ordering with respect to the property that \( L \subseteq Se_1 \cap Se_2 \) and if \( J \) is the \( \mathcal{J} \)-class (= \( \mathcal{D} \)-class) of \( S \) containing \( L \), then

(a) \( J \) is a regular \( \mathcal{J} \)-class of \( S \), and

(b) if \( J^0 \approx \mathcal{M}^0(G; I, \Lambda, P) \) is a Rees matrix representation of \( J^0 \) with maximal subgroup \( G \), index sets \( I \) and \( \Lambda \) (for the \( \mathcal{R} \) and \( \mathcal{L} \)-classes, respectively), and sandwich matrix \( P \), if \( \Lambda' \) is the subset of \( \Lambda \) which corresponds to the set of \( \mathcal{L} \)-classes of \( J \) which satisfy the above maximality condition, and if \( \mathcal{M}^0(G; I, \Lambda', P') \) is the correspondingly restricted Rees matrix semigroup, then \( P' \) has a right inverse as a matrix over \( kG \).

Proof. By the corollary to Lemma 1, there is some \( \gamma = \gamma^2 \in kSe_1 \cap kSe_2 \) such that \( kSe_1 \cap kSe_2 = kS\gamma \). We observe also that \( kSe_1 \cap kSe_2 = k(Se_1 \cap Se_2) \). Suppose \( Se_1 \cap Se_2 \neq \emptyset \), and let \( L \) be an \( \mathcal{L} \)-class of \( S \) maximal with respect to \( L \subseteq Se_1 \cap Se_2 \). Let \( U = \{ x \in S : Sx \nsubseteq L \} \). Then \( U \) and \( U \cap L \) are left ideals of \( S \), \( U \cap L = \emptyset \), and the maximality of \( L \) implies that \( Se_1 \cap Se_2 \subseteq U \cap L \).

Now one can write \( \gamma = \gamma_L + \gamma_U \) where \( \gamma_L \) is a \( k \)-linear combination of elements of \( L \) and \( \gamma_U \) is a \( k \)-linear combination of elements of \( U \). If \( s \in L \subseteq Se_1 \cap Se_2 \), then \( s = s\gamma = s\gamma_L + s\gamma_U \), where \( s\gamma_L \) is a \( k \)-linear combination of elements of \( L^2 \) and \( s\gamma_U \) is a \( k \)-linear combination of elements of \( U \). Since the elements of \( S \) are linearly independent over \( k \), this implies that \( L \subseteq L^2 \). If \( J \) is the \( \mathcal{J} \)-class containing \( L \), then \( J \cap J^2 \neq \emptyset \). Hence \( J \) is regular, and thus \( L \) contains an idempotent. Since \( Se_1 \cap Se_2 \) is a maximal \( \mathcal{L} \)-class as above, we have that \( Se_1 \cap Se_2 \) is generated by idempotents.

Let \( J^0 \approx \mathcal{M}^0(G; I, \Lambda, P) \); let \( \{ L_\lambda : \lambda \in \Lambda \} \) be the \( \mathcal{L} \)-classes of \( J \); and let \( \{ L_\lambda : \lambda \in \Lambda' \} \), \( \Lambda' \subseteq \Lambda \) be the set of \( \mathcal{L} \)-classes of \( J \) maximal in the \( \mathcal{L} \)-class ordering with respect to \( L_\lambda \subseteq Se_1 \cap Se_2 \). Then we let \( P' \) be the corresponding
\[ |\Lambda'| \times |I| \] submatrix of the \[ |\Lambda| \times |I| \] matrix \( P \) and see that the Rees matrix semigroup \( \mathcal{M}(G; I, \Lambda', P') \) is isomorphic to \( V^0 \) where \( V' = \bigcup \{ L_1 : \lambda \in \Lambda' \} \).

Let \( W = \{ x \in S : Sx \cap V = \emptyset \} \). Then \( W \) and \( W \cup V \) are left ideals of \( S \), \( W \cap V = \emptyset \), and \( Se_1 \cap Se_2 \subseteq W \cup V \). We can write \( \gamma = \gamma_V + \gamma_W \) with \( \gamma_V \) (respectively \( \gamma_W \)) a \( k \)-linear combination of elements of \( V \) (respectively \( W \)). If we write \( \gamma_V \) as an \( |I| \times |\Lambda'| \) matrix \( Q \) over \( kG \), then the fact that \( s\gamma_V = s\gamma_W = s - s\gamma_W \) for all \( s \in V \), and \( s\gamma_W \in kW \), implies that the matrix product \( P'Q \) is a \[ |\Lambda'| \times |\Lambda'| \] identity matrix.

**Corollary 1.** In the situation of (ii) above, it must be that \( |I| \geq |\Lambda'| \).

**Proof.** If we extend the augmentation homomorphism \( kG \rightarrow k \), given by \( gh \rightarrow 1 \) for all \( g \in G \), to matrices over \( kG \), then since \( P'Q \) is the identity, the rank of the image of \( P' \) must be \( |\Lambda'| \), which implies that \( |I| \geq |\Lambda'| \).

**Corollary 2.** No two rows of \( P' \) have zero and nonzero entries in exactly the same locations.

**Proof.** If \( P' \) had two such rows, then the rank of the image of \( P' \) under the augmentation homomorphism would be less than \( |\Lambda'| \), impossible.

**Corollary 3.** If \( S \) is a union of groups semigroup, if \( e_1 \) and \( e_2 \in S \) are idempotents, and if \( kSe_1 + kSe_2 \) is \( kS \)-projective, then if \( Se_1 \cap Se_2 \neq \emptyset \), the \( \mathcal{L} \)-classes \( L \) of \( S \) maximal with respect to \( L \subseteq Se_1 \cap Se_2 \) all lie in distinct \( J \)-classes.

**Proof.** If \( S \) is a union of groups then all the entries of \( P \) and \( P' \) above must be nonzero. Thus by Corollary 2, \( P' \) has only one row, that is, \( |\Lambda'| = 1 \), which says that only one \( \mathcal{L} \)-class \( L \) of \( J \) is maximal with respect to \( L \subseteq Se_1 \cap Se_2 \).

**Theorem 2.** Let \( S \) be a union of groups monoid such that the \( J \)-classes are linearly ordered. If \( kS \) is left hereditary, then the intersection of any two principal left ideals of \( S \) is either empty or itself principal.

If, in addition, the ring \( k \) is noetherian (e.g., if \( k \) is a field), then \( kS \) is right hereditary also, so that the intersection of any two principal right ideals is either empty or again principal.

**Proof.** The last part of the theorem follows from the first part by observing that \( k \) noetherian and \( S \) finite imply that \( kS \) is noetherian and that right and left global dimension are equal for noetherian rings (see, e.g., [3]).

To prove the first part we see first that since \( S \) is a union of groups, principal left ideals have idempotent generators. Let \( e_1, e_2 \in S \) be idempotents. Then \( kSe_1 + kSe_2 \) is \( kS \)-projective since \( kS \) is left hereditary. Hence there exists an idempotent \( \gamma \in kSe_1 \cap kSe_2 \) such that \( kSe_1 \cap kSe_2 = kSy \).
Also, by Corollary 3 above, if $S e_1 \cap S e_2 \neq \emptyset$, the $L$-classes $L$ of $S$ maximal with respect to $L \subseteq S e_1 \cap S e_2$ lie in distinct $\mathcal{J}$-classes of $S$. Then the proof is concluded by the following lemma.

**Lemma.** If $S$ is a union of groups monoid whose $\mathcal{J}$-classes are linearly ordered and if $I \subseteq S$ is a left ideal such that the $L$-classes $L$ of $S$ maximal with respect to $L \subseteq I$ lie in distinct $\mathcal{J}$-classes of $S$, then $k I$ has an idempotent generator if and only if $I$ is principal.

**Proof.** If $I$ is a principal left ideal of $S$ it has an idempotent generator in $S$, which will also serve as an idempotent generator for $k I$ in $k S$.

Now suppose we have $I$ and an idempotent $y = y^2 \in k I$ such that $k I = k S y$. Suppose that $L_1$ and $L_2$ are distinct $L$-classes of $S$ maximal with respect to $L_i \subseteq I$. Then $L_1$ and $L_2$ lie in distinct $\mathcal{J}$-classes of $S$, say $L_i = J_i$, $i = 1, 2$, and $J_1 \supseteq J_2$ in the $\mathcal{J}$-class ordering. Let $I' = I \setminus (L_1 \cup L_2)$. Then $I'$ is a left ideal of $S$. We can write $y = y_1 + y_2 + y'$ where $y_i$ is a $k$-linear combination of elements of $L_i$ for $i = 1, 2$, and $y'$ is a $k$-linear combination of elements of $I'$. Let $e_i \in L_i$ be idempotents for $i = 1, 2$. Now $e_1 e_2 \mathcal{J} e_2 e_1 \mathcal{J} e_2$ since $S$ is a union of groups (see [2, Theorem 4.4]). Hence we know that $e_1 e_2 L e_2$ so that since $J_1$ and $J_2$ are completely simple, $e_1 L_2 \subseteq L_2$ and $e_1 L_i \subseteq L_i$ for $i = 1, 2$. The maximality of $L_2$ and the fact that $e_2 e_1 \mathcal{J} e_2$ gives $e_2 L_1 \cap L_2 = \emptyset$.

Now $e_1 = e_1 y = e_1 y_1 + e_1 y_2 + e_1 y'$ implies that $e_1 y_1 = e_1$, $e_1 y_2 = 0$, and $e_1 y' = 0$ since elements of $S$ are linearly independent over $k$. Thus if $y_2 = \sum_{v \in L_2} m_v y$, $m_v \in k$, this will imply that $\sum_{v \in L_2} m_v = 0$. But also $e_2 = e_2 y = e_2 y_1 + e_2 y_2 + e_2 y'$ implies that $e_2 y_1 = e_2$, since $e_2 y_2$ is a $k$-linear combination of elements of $L_2$ while $e_2 y_1 + e_2 y'$ is a $k$-linear combination of elements of $I \setminus L_2$. However, $e_2 = e_2 y_1$ implies that $\sum_{v \in L_2} m_v = 1$, contradiction. Thus it is impossible that there be two distinct $L$-classes of $S$ maximal with respect to containment in $I$. Thus there is a unique such $L$-class, say $L_1$, which then says that $I = S e_1$.

**References**