PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 37, Number 2, February 1973

# A CHARACTERIZATION OF LOCAL COMPACTNESS

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ABSTRACT. In this paper we show that the cluster set of a filter in a Hausdorff space $X$ is a continuous function of the filter if and only if $X$ is locally compact.

Our result should be compared with that of Wyler [2] who has shown that a Hausdorff space is regular if and only if the limit of a convergent filter is a continuous function of the filter. Wyler uses a different topology on his space of filters.

For a filter $\mathscr{F}$ in a topological space $X$ the cluster set of $\mathscr{F}$ is $\bigcap \{F^-: F \in \mathscr{F}\}$. We will denote the cluster set of $\mathscr{F}$ by $\alpha(\mathscr{F})$.

For a set $X$ and subset $U$ of $X$, $P(X)$ is the set of all nonempty subsets of $X$ and $R(U, X)$ is the family of all nonempty subsets of $X$ which intersect $U$. If $X$ is a topological space, the *lower semifinite* (lsf) topology on $P(X)$ has as a subbasis all sets of the form $R(U, X)$ where $U$ is open in $X$.

We will denote all filters of a topological space $X$ which have nonempty cluster sets by $X^{\#}$. The filter space $X^{\#}$ will be assumed to carry the topology which has as a subbasis all sets of the form

$$U^{\#} = \{\mathscr{F} \in X^{\#} : F \cap U \neq \varnothing \text{ for all } F \text{ in } \mathscr{F}\},$$

where $U$ is open in $X$.

The reader is referred to Kelley [1] for definitions and results not given here.

We can now state the result which we propose to prove as follows.

THEOREM. *Let $X$ be a Hausdorff space and $P(X)$ have the lsf topology. Then the cluster set function $\alpha : X^{\#} \to P(X)$ is continuous if and only if $X$ is locally compact.*

PROOF. Assume $X$ is locally compact. Let $\mathscr{F}_0$ be in $X^{\#}$ and $R(V, X)$, where $V$ is open, be a subbasic neighborhood of $\alpha(\mathscr{F}_0)$. Then for some $p$ in $\alpha(\mathscr{F}_0)$, $p$ is also in $V$. Hence there is a compact neighborhood $U$ of $p$ contained in $V$.

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Received by the editors January 31, 1972.

*AMS (MOS) subject classifications* (1970). Primary 54D45, 54A99; Secondary 54B20, 54C60.

*Key words and phrases*. Local compactness, space of filters, cluster set of a filter, lower semifinite topology.

Consider the neighborhood of $\mathscr{F}_0$, $U^{\#}$, and let $\mathscr{F}$ be in this neighborhood. Then for each $F$ in $\mathscr{F}$, $F \cap U \neq \varnothing$ so that a filter $\mathscr{F}_U$ is generated by the collection $\{F \cap U : F \in \mathscr{F}\}$. Since $U$ is compact, $\mathscr{F}_U$ must have a cluster point $q$ in $U$. But $\mathscr{F}$ is coarser than $\mathscr{F}_U$, so $q$ is also a clustei point of $\mathscr{F}$. Thus $\alpha(\mathscr{F}) \cap U \neq \varnothing$ so $\alpha(\mathscr{F}) \in R(V, X)$. Therefore $\alpha$ is continuous.

Assume $X$ is not locally compact. There is then a point $p$ in $X$ such that no neighborhood of $p$ is compact. Hence for every neighborhood $U$ of $p$, there is a filter $\mathscr{F}_U$ in $U$ which has no cluster point.

Let $\mathscr{F}_0$ be the filter of all supersets of $\{p\}$. If $W^{\#}$, where $W$ is open, is a subbasic neighborhood of $\mathscr{F}_0$, then in particular $\{p\} \cap W \neq \varnothing$ so $W$ is a neighborhood of $p$. Note since $X$ is Hausdorff, $\alpha(\mathscr{F}_0) = \{p\}$.

Now let $R(V, X)$, where $V$ is open, be a subbasic neighborhood of $\alpha(\mathscr{F}_0)$, so that $V$ is a neighborhood of $p$. Consider for each neighborhood $U$ of $p$, the filter $\mathscr{G}_U = \{F \cup (X - V) : F \in \mathscr{F}_U\}$ which has cluster set $\alpha(\mathscr{G}_U) = X - V$. If $W^{\#}$ is a subbasic neighborhood of $\mathscr{F}_0$, $\mathscr{G}_U \in W^{\#}$ for $U \subset W$. Thus the net of filters $\mathscr{G}_U$ converges to $\mathscr{F}_0$. But $\alpha(\mathscr{G}_U)$ does not belong to $R(V,X)$ for any $U$. Therefore $\alpha : X^{\#} \to P(X)$ is not continuous.

## References

**1.** J. L. Kelley, *General topology*, Van Nostrand, Princeton, N.J., 1955. MR **16,** 1136.

**2.** Oswald Wyler, *A characterization of regularity in topology*, Proc. Amer. Math. Soc. **29** (1971), 588–590. MR **43** #6865.

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