A CHARACTERIZATION OF LOCAL COMPACTNESS

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Abstract. In this paper we show that the cluster set of a filter in a Hausdorff space $X$ is a continuous function of the filter if and only if $X$ is locally compact.

Our result should be compared with that of Wyler [2] who has shown that a Hausdorff space is regular if and only if the limit of a convergent filter is a continuous function of the filter. Wyler uses a different topology on his space of filters.

For a filter $\mathcal{F}$ in a topological space $X$ the cluster set of $\mathcal{F}$ is $\bigcap \{F^- : F \in \mathcal{F}\}$. We will denote the cluster set of $\mathcal{F}$ by $\alpha(\mathcal{F})$.

For a set $X$ and subset $U$ of $X$, $P(X)$ is the set of all nonempty subsets of $X$ and $R(U, X)$ is the family of all nonempty subsets of $X$ which intersect $U$. If $X$ is a topological space, the lower semifinite (lsf) topology on $P(X)$ has as a subbasis all sets of the form $R(U, X)$ where $U$ is open in $X$.

We will denote all filters of a topological space $X$ which have nonempty cluster sets by $X^\#$. The filter space $X^\#$ will be assumed to carry the topology which has as a subbasis all sets of the form

$$U^\# = \{\mathcal{F} \in X^\#: F \cap U \neq \emptyset \text{ for all } F \text{ in } \mathcal{F}\},$$

where $U$ is open in $X$.

The reader is referred to Kelley [1] for definitions and results not given here.

We can now state the result which we propose to prove as follows.

Theorem. Let $X$ be a Hausdorff space and $P(X)$ have the lsf topology. Then the cluster set function $\alpha: X^\# \to P(X)$ is continuous if and only if $X$ is locally compact.

Proof. Assume $X$ is locally compact. Let $\mathcal{F}_0$ be in $X^\#$ and $R(V, X)$, where $V$ is open, be a subsbasic neighborhood of $\alpha(\mathcal{F}_0)$. Then for some $p$ in $\alpha(\mathcal{F}_0)$, $p$ is also in $V$. Hence there is a compact neighborhood $U$ of $p$ contained in $V$.
Consider the neighborhood of \( F_0, U^# \), and let \( F \) be in this neighborhood. Then for each \( F \) in \( F \), \( F \cap U \neq \emptyset \) so that a filter \( F_U \) is generated by the collection \( \{ F \cap U : F \in F \} \). Since \( U \) is compact, \( F_U \) must have a cluster point \( q \) in \( U \). But \( F \) is coarser than \( F_U \), so \( q \) is also a cluster point of \( F \). Thus \( \alpha(F) \cap U \neq \emptyset \) so \( \alpha(F) \in R(V, X) \). Therefore \( \alpha \) is continuous.

Assume \( X \) is not locally compact. There is then a point \( p \) in \( X \) such that no neighborhood of \( p \) is compact. Hence for every neighborhood \( U \) of \( p \), there is a filter \( F_U \) in \( U \) which has no cluster point.

Let \( F_0 \) be the filter of all supersets of \( \{ p \} \). If \( W^# \), where \( W \) is open, is a subbasic neighborhood of \( F_0 \), then in particular \( \{ p \} \cap W \neq \emptyset \) so \( W \) is a neighborhood of \( p \). Note since \( X \) is Hausdorff, \( \alpha(F_0) = \{ p \} \).

Now let \( R(V, X) \), where \( V \) is open, be a subbasic neighborhood of \( \alpha(F_0) \), so that \( V \) is a neighborhood of \( p \). Consider for each neighborhood \( U \) of \( p \), the filter \( G_U = \{ F \cap (X - V) : F \in F_U \} \) which has cluster set \( \alpha(G_U) = X - V \). If \( W^# \) is a subbasic neighborhood of \( F_0 \), \( G_U \in W^# \) for \( U \subseteq W \). Thus the net of filters \( G_U \) converges to \( F_0 \). But \( \alpha(G_U) \) does not belong to \( R(V, X) \) for any \( U \). Therefore \( \alpha : X^# \to P(X) \) is not continuous.

REFERENCES


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