A CHARACTERIZATION OF LOCAL COMPACTNESS
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Abstract. In this paper we show that the cluster set of a filter in a Hausdorff space $X$ is a continuous function of the filter if and only if $X$ is locally compact.

Our result should be compared with that of Wyler [2] who has shown that a Hausdorff space is regular if and only if the limit of a convergent filter is a continuous function of the filter. Wyler uses a different topology on his space of filters.

For a filter $\mathcal{F}$ in a topological space $X$ the cluster set of $\mathcal{F}$ is $\{F^c : F \in \mathcal{F}\}$. We will denote the cluster set of $\mathcal{F}$ by $\alpha(\mathcal{F})$.

For a set $X$ and subset $U$ of $X$, $P(X)$ is the set of all nonempty subsets of $X$ and $R(U, X)$ is the family of all nonempty subsets of $X$ which intersect $U$. If $X$ is a topological space, the lower semifinite (lsf) topology on $P(X)$ has as a subbasis all sets of the form $R(U, X)$ where $U$ is open in $X$.

We will denote all filters of a topological space $X$ which have nonempty cluster sets by $X$#. The filter space $X^#$ will be assumed to carry the topology which has as a subbasis all sets of the form

$$U^# = \{\mathcal{F} \in X^# : F \cap U \neq \emptyset \text{ for all } F \in \mathcal{F}\},$$

where $U$ is open in $X$.

The reader is referred to Kelley [1] for definitions and results not given here.

We can now state the result which we propose to prove as follows.

Theorem. Let $X$ be a Hausdorff space and $P(X)$ have the lsf topology. Then the cluster set function $\alpha : X^# \to P(X)$ is continuous if and only if $X$ is locally compact.

Proof. Assume $X$ is locally compact. Let $\mathcal{F}_0$ be in $X^#$ and $R(V, X)$, where $V$ is open, be a subbasic neighborhood of $\alpha(\mathcal{F}_0)$. Then for some $p$ in $\alpha(\mathcal{F}_0)$, $p$ is also in $V$. Hence there is a compact neighborhood $U$ of $p$ contained in $V$.

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Consider the neighborhood of $\mathcal{F}_0, U^\#$, and let $\mathcal{F}$ be in this neighborhood. Then for each $F$ in $\mathcal{F}, F \cap U \neq \emptyset$ so that a filter $\mathcal{F}_U$ is generated by the collection $\{F \cap U : F \in \mathcal{F}\}$. Since $U$ is compact, $\mathcal{F}_U$ must have a cluster point $q$ in $U$. But $\mathcal{F}$ is coarser than $\mathcal{F}_U$, so $q$ is also a cluster point of $\mathcal{F}$. Thus $\alpha(\mathcal{F}) \cap U \neq \emptyset$ so $\alpha(\mathcal{F}) \in R(V, X)$. Therefore $\alpha$ is continuous.

Assume $X$ is not locally compact. There is then a point $p$ in $X$ such that no neighborhood of $p$ is compact. Hence for every neighborhood $U$ of $p$, there is a filter $\mathcal{F}_U$ in $U$ which has no cluster point.

Let $\mathcal{F}_0$ be the filter of all supersets of $\{p\}$. If $W^\#$, where $W$ is open, is a subbasic neighborhood of $\mathcal{F}_0$, then in particular $\{p\} \cap W \neq \emptyset$ so $W$ is a neighborhood of $p$. Note since $X$ is Hausdorff, $\alpha(\mathcal{F}_0) = \{p\}$.

Now let $R(V, X)$, where $V$ is open, be a subbasic neighborhood of $\alpha(\mathcal{F}_0)$, so that $V$ is a neighborhood of $p$. Consider for each neighborhood $U$ of $p$, the filter $\mathcal{G}_U = \{F \cap (X - V) : F \in \mathcal{F}_U\}$ which has cluster set $\alpha(\mathcal{G}_U) = X - V$. If $W^\#$ is a subbasic neighborhood of $\mathcal{F}_0$, $\mathcal{G}_U \in W^\#$ for $U \subseteq W$. Thus the net of filters $\mathcal{G}_U$ converges to $\mathcal{F}_0$. But $\alpha(\mathcal{G}_U)$ does not belong to $R(V, X)$ for any $U$. Therefore $\alpha : X^\# \rightarrow P(X)$ is not continuous.

REFERENCES