WEYL'S LEMMA FOR POINTWISE SOLUTIONS
OF ELLIPTIC EQUATIONS

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Abstract. We prove that pointwise, $L_1$ solutions of second order elliptic partial differential equations are classical solutions.

0. Introduction. A consequence of Weyl's lemma for second order elliptic partial differential equations is that every $L_1$ (Lebesgue class) weak solution is a classical solution, i.e., if, for

$$L = \sum_{i,j=1}^{K} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{K} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

and

$$L^* = \sum_{i,j=1}^{K} \partial^2 \frac{\partial}{\partial x_i \partial x_j} a_{ij}(x) - \sum_{i=1}^{K} \partial \frac{\partial}{\partial x_i} b_i(x) + c(x),$$

we have $\int_{\Omega} uL^* \phi = \int_{\Omega} f \phi$ for all $\phi$ in $C_0^\infty(\Omega)$, then $Lu = f$ in $\Omega$.

The differentiability of the coefficients of $L$, required for the definition of $L^*$, is not intrinsic in view of the maximum principle and the Dirichlet problem (in the case $c(x) \geq 0$). Our aim is to employ a notion of generalized solution of $Lu = f$ which bypasses the adjoint operator and thereby establish an analogue of Weyl's lemma when the coefficients are in a Hölder-$\alpha$ class. Definitions and the statement of the theorem follow in the next section.

1. Preliminaries. We shall work in $K$-dimensional Euclidean space, $\mathbb{R}^K$, $3 \leq K$, and shall use the following notation: $x = (x_1, \ldots, x_K)$ and $B(x, r) =$ the open $K$-ball centered at $x$ with radius $r$. $\Omega$ will denote an open set in $\mathbb{R}^K$; $|E|$, the Lebesgue measure of $E$; $\partial E$, the boundary of $E$; subscripted $e$, constants which depend only on the operator (1) and $K$; and $\omega_K$, the surface area of $\partial B(0, 1)$.

A function $u$, which is in $L_1$ in a neighborhood of $x$, is said to be in
For $\beta=2$, $P(y)=\gamma_0+\sum_{i=1}^{K} \gamma_i y_i+2^{-1} \sum_{i,j=1}^{K} \gamma_{ij} y_i y_j$, where $\gamma_{ij}=\gamma_{ji}$; we define the generalized partial derivatives $D_i u(x)$, $D_{ij} u(x)$ to be $\gamma_i$, $\gamma_{ij}$ respectively. Also we can redefine $u$ at $x$, if necessary, so that $u(x)=\gamma_0$.

(The class $t_\beta$ was defined in [1] and has since appeared in [2] and [6].)

For $u$ in $t_2(x)$, we say that $u$ is a $t_2$ solution of $Lu=f$ at $x$ if

$$r^{-K} \int_{B(x, r)} |u(x+y) - P(y)| \, dy = o(r^\beta) \quad \text{as } r \to 0.$$  

We say that $u$ is a $t_2$ solution of $Lu=f$ in $\Omega$ if $u$ is a $t_2$ solution at each point in $\Omega$. $u$ is said to be a classical solution if $u \in C^2(\Omega)$. (It should be noted that no restrictions have been placed on the operator $L$ up to this point.)

We shall assume the operator (1) to be elliptic and have $\alpha$-Hölder continuous coefficients, $0<\alpha\leq 1$; with this, (1) is uniformly elliptic on compact subsets of $\Omega$.

**Theorem.** If $u$ is an $L_1$, $t_2$ solution of $Lu=f$ in $\Omega$, then $u$ is a classical solution.

**Remarks.** No assumption has been made on the integrability of the generalized derivatives $D_i u(x)$, $D_{ij} u(x)$ when considered as functions on $\Omega$. In fact, see [2], there are simple examples of functions in $t_2(x)$, $x$ in $(-1, 1)$ such that $D_i u(x)$, $D_{ij} u(x)$ are not in $L_1$ on compact subsets of $(-1, 1)$; note that the $o(r^\beta)$ in (2) is not assumed to be uniform in $x$.

The special case of $u$ in $L_\infty$, $f=0$, and $c(x)\leq 0$, has been established in [2]. The case for general $f$, $\alpha$-Hölder continuous, is immediate. The restriction on $c(x)$ is not essential and while a nontrivial argument is necessary for unrestricted $c(x)$, the techniques are essentially those of [2, §§1-4], and those of this paper. We therefore maintain $c(x)\leq 0$.

As in [2], we only need assume that $u$ is a $t_2$ solution of $Lu=f$ almost everywhere and in $t_2$ everywhere in $\Omega$. We shall omit this extension.

It is not clear that the Hölder-$\alpha$ condition is best possible. However, our result is best possible with respect to the notion of $t_2$ solution in that the conclusion fails if the absolute values in (2) are not required, i.e., $u(x)\equiv x_1|x|^{-K}$ satisfies, with $P(y)\equiv 0$ for $x=0$,

$$r^{-K} \int_{B(x, r)} |u(x+y) - P(y)| \, dy = o(r^\beta) \quad \text{as } r \to 0$$

for all $x$; that is, $x_1|x|^{-K}$ has generalized Laplacian equal to zero everywhere and is clearly not harmonic at 0.

A forthcoming paper in the Ann. Scuola Norm. Sup. Pisa by Hager and Ross deals with elliptic equations in divergence form with $\alpha$-Hölder coefficients; the notion of weak solution is considered vis-à-vis the notion of pointwise solution employed in this paper.

This work is an extension of some of the concepts developed by V. L. Shapiro in [6].

We also wish to thank the referee for his many helpful suggestions.

2. Fundamental lemmas. The first three lemmas were established in [2]. By a portion $P$ of a set $Z$, we shall mean a nonempty intersection of $Z$ with an open ball.

**Lemma 1** [2, Lemma 2]. If $u$ is in $t_2(x)$ for all $x$ in $Z$ and if $Z$ is closed and nonempty, then there is a portion $P$ of $Z$ such that $u$ is continuous in $P$ in the relative topology, i.e., if $x_1 \in P$, then given $\varepsilon > 0$, there is a $\delta > 0$ such that for $x_2 \in P$ and $|x_1 - x_2| < \delta$, $|u(x_1) - u(x_2)| < \varepsilon$.

**Lemma 2** (see [2, proof of Theorem 1]). If $u$ is in $t_2(x)$ for all $x$ in $Z$, closed and nonempty, then there is a portion $P$ of $Z$ and positive constants $M$ and $r_0$ such that for all $x$ in $P$ and $0 < r < r_0$,

$$|B(0, r)|^{-1} \int_{B(0, r)} |u(x + y) - u(x)| \, dy \leq M. \quad (4)$$

**Lemma 3** [2, Theorem 1]. If $u$ is an $L_\infty$, $t_2$ solution of $Lu = 0$ ($c(x) \leq 0$) in $\Omega$, then $u$ is a classical solution.

**Lemma 4** (J. Serrin, [3, p. 300]). There exist functions $K_+(x, y)$ and $K_-(x, y)$ for $|x| \leq r$, $|y| = r$, $x \neq y$, and $r \leq 1$ having the following properties:

(i) Considered as functions of $x$ for fixed $y$,

$$LK_+ \geq 0 \quad \text{and} \quad LK_- \leq 0. \quad (5)$$

(ii) For any point $y_0$, $|y_0| = r$, and any continuous function $g(y)$,

$$\lim_{z \to y_0} \int_{t \in B(0, r)} K_\pm(x, y) g(y) \, dS_y(y) = g(y_0)$$

where $dS_y(y)$ is the natural surface area element.

(iii) There are positive constants $e_1$ and $e_2$ such that, for $|x| \leq r/3$,

$$K_+(x, y) \geq e_1 r^{1-K} \quad \text{and} \quad K_-(x, y) \leq e_2 r^{1-K}. \quad (7)$$

**Lemma 5**. If $Lu = 0$ in $B(x_0, r)$, then

$$|u(x_0)| \leq e_3 |B(x_0, r)|^{-1} \int_{B(x_0, r)} |u(x)| \, dx. \quad (8)$$
Proof. As in [4], \( u(x) = \int_{\partial B(x_0, \rho)} u(y) \, d\omega(x, y) \) where \( d\omega(x, y) \) is a nonnegative Borel measure on \( \partial B(x_0, \rho) \), with \( \rho < r \). Thus
\[
|u(x)| \leq \int_{\partial B(x_0, \rho)} |u(y)| \, d\omega(x, y).
\]
Form
\[
u_i(x) = \int_{\partial B(x_0, \rho)} K_i(x, y) |u(y)| \, dS_{\rho}(y).
\]
By (5), (6), and the maximum principle,
\[
\int_{\partial B(x_0, \rho)} |u(y)| \, d\omega(x, y) \leq u_-(x) \quad \text{for } x \in B(x_0, \rho),
\]
and by (7),
\[
u_-(x) \leq \varepsilon_2 \rho^{1-K} \int_{\partial B(x_0, \rho)} |u(y)| \, dS_{\rho}(y) \quad \text{for } |x - x_0| \leq \rho/3.
\]
Hence
\[
|u(x_0)| \leq \varepsilon_2 \rho^{1-K} \int_{\partial B(x_0, r)} |u(y)| \, dS_{\rho}(y).
\]
Thus by integrating
\[
\frac{r^{1-K}}{K} |u(x_0)| \leq \varepsilon_2 \int_{B(x_0, r)} |u(y)| \, dy
\]
which yields (8).

Proof of the Theorem (\( c(x) \leq 0 \)). Let \( Z \) be the set of discontinuities of \( u \); \( \overline{Z} \) is its closure. If \( Z \) is empty, then by Lemma 3, \( u \) is a classical solution. Assuming, therefore, that \( \overline{Z} \neq \emptyset \), we have, by Lemmas 1 and 2, a portion \( P \) of \( Z \) in which \( u \) is continuous in the relative topology and satisfies (4) for \( x \) in \( P \). Let \( x_0 \in P \). If we can show that there is a \( \delta > 0 \) such that \( u \) is bounded in \( |x - x_0| < \delta \), then \( u \) is a classical solution in \( |x - x_0| < \delta \).

Consequently, \( x_0 \) is not in \( Z \); therefore \( x_0 \) is not in \( P \), which is a contradiction based on the assumption that \( \overline{Z} \neq \emptyset \).

We can choose \( \delta = \min(r_0/2, r_1/2) \), where \( r_0 \) is given in Lemma 2 and where \( r_1 \) is selected so that

(i) \( 0 < r_1 \),

(ii) \( \{|x_0 - x| < r_1, x \in Z, |u(x) - u(x_0)| \leq 1\} \subseteq P \).

For \( |x - x_0| < \delta \), \( x \) in \( Z \),
\[
|u(x)| \leq |u(x) - u(x_0)| + |u(x_0)| \leq |u(x_0)| + 1.
\]

For \( |x - x_0| < \delta \), \( x \) not in \( Z \), there is a point \( x^* \) in \( P \) and \( 0 < \rho < \delta \) such that \( x^* \in \partial B(x, \rho) \) while \( P \cap B(x, \rho) = \emptyset \). Thus \( Lu = 0 \) in \( B(x, \rho) \). Hence,
by (8),

$$|u(x)| \leq e_3 |B(x, \rho)|^{-1} \int_{B(x, \rho)} |u(y)| \, dy$$

$$\leq e_3 2^{2K} |B(x^*, 2\rho)|^{-1} \int_{B(x^*, 2\rho)} |u(y)| \, dy.$$  

Since $\rho < \delta < r_0/2$, $2\rho < r_0$, and $x^* \in P$,

$$\leq e_3 2^{2K} [u(x^*)] + M.$$  

Since $|x_0 - x^*| \leq |x_0 - x| + |x - x^*| \leq \delta + \rho < r_1$, $|u(x^*)| \leq |u(x_0)| + 1$, which gives $|u(x)| \leq e_3 2^{2K} [u(x_0)| + M + 1]$, completing the proof.

**Bibliography**


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