A NECESSARY CONDITION IN THE CALCULUS OF VARIATIONS

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ABSTRACT. A necessary condition that an extremal be a relative minimum is derived. The condition includes and may be stronger than the Legendre-Hadamard condition.

By considering a slightly more general variation than Hadamard we will obtain a necessary condition in the calculus of variations which may be stronger than the Legendre-Hadamard condition [1, p. 253] and [2, p. 11].

If \( \phi \in L(E, F) \) and \( p \in E \) let \([p, \phi]=\phi(p)\) where \( E \) and \( F \) are vector spaces. If \( g \in L(E, L(E, F)) \) and if \( h, k \in E \) let \( ghk=[k, [h, g]] \) and \( gh^{(2)}=ggh \).

Let \( e_a \in R^r, a=1, \ldots, r \), and \( E_i \in R^N, i=1, \ldots, N \), be defined by
\[
e_a = (\delta_1^a, \ldots, \delta_r^a) \quad \text{and} \quad E_i = (\delta_1^i, \ldots, \delta_N^i).
\]

Let \( e^p \in L(R^r, R)=R^r \) be defined by \( e^p e_a = \delta_a^p \). Let \( e^p E_i \) be that element of \( L(R^r, R^N) \) defined by \( (e^p E_i)e_a = \delta_a^i E_i \). If \( \lambda \in R_+ \) and \( \xi \in R^N \) then \( (\lambda \xi)e_a = (\lambda \delta_a^b \xi^b E_i) = \lambda \delta_a^i \xi^i E_i \) (tensor convention for summation). If \( \phi \in C''(L(R^r, R^N), F) \) we write \( \phi^1 \) for \( \phi e^1 E_i \) and \( \phi^p_{ij} \) for \( \phi^p e^p E_i E_j \). Thus if \( \lambda = \lambda \delta_e e, \xi = \xi^i E_i \in R^N \),
\[
\phi^p_{ij} \lambda \delta_e^i \xi^j = \phi^p e^p E_i E_j, \lambda \delta_e^i \xi^j = \phi^p (\lambda \xi)^{(2)}.
\]

Let \( G \) be a bounded domain in \( R^r \) and \( f \in C'(G \times R^N \times L(R^r, R^N), R) \). Let \( C'=C'(G', R^N) \) and if \( z \in C' \) let \( \|z\| = \text{sup} \{\|z(x)\| + \|z'(x)\| \mid x \in G\} \). If \( z \in C' \) let \( I_f(z)=\int_G f(x, z(x), p(x)) \, dx \) where \( p=z' \). Suppose that \( I_f(z) \leq I_f(z+\zeta) \) whenever \( \zeta \in C' \), support \( \zeta \subset G \) and \( \|\zeta\| \) is small enough. Then \( \phi''(0) \geq 0 \) where \( \phi(\lambda)=I(z+\lambda \zeta) \) whenever \( \zeta \in C' \) and support \( \zeta \subset G \). Thus
\[
\int_G \left\{ f_{zz}(P)\xi^{(2)}(x) + 2f_{zp}(P)\xi(x)\zeta(x) + f_{pp}(P)\zeta^{(2)}(x) \right\} \, dx \geq 0
\]
where \( P = (x, z(x), p(x)) \). Let \( x_0 \in G \), let \( n \) be so large that
\[
-\frac{1}{n}(-x_0 + \text{support } \zeta) \subset -x_0 + G
\]
and let \( \zeta_n \) be defined on \( G \) with support \( \zeta_n \subset x_0 + n(-x_0 + \text{support } \zeta) \) by
\[
\zeta_n(x_0 + y) = \zeta(x_0 + ny)^n. 
\]
Evidently support \( \zeta_n \subset G \), \( \zeta_n \in C' \), and
\[
\zeta_n(x_0 + y) = \zeta(x_0 + ny). 
\]
Using the continuity of \( f' \) and letting \( n \to \infty \) we see that
\[
f_{pp}(x_0, z(x_0), p(x_0)) \int_G \zeta_n''(x) \, dx \geq 0.
\]
By approximations we obtain

**Lemma.** If \( z \) is a weak relative minimum for \( I \) then
\[
f_{pp}(x_0, z(x_0), p(x_0)) \int_G \zeta''(x) \, dx \geq 0
\]
for all Lipschitzian \( \zeta \) with support \( \zeta \subset G \).

Now let \( \xi_1, \ldots, \xi_n \in \mathbb{R}^N \) and \( \lambda_1, \ldots, \lambda_n \) be a linearly independent set in \( \mathbb{R}^n \). Let \( y^a = \lambda_\alpha(x^\alpha - x^{\alpha}_0) \) and \( r = \|y\| \). Let \( h > 0 \) be so small that \( x \in G \) if \( r \leq h \). If \( i = 1, \ldots, N \) let \( w^i \) be defined on \( G' = \{y| x \in G\} \) with support \( w^i \) contained in \( r \leq h \) by \( w^i(y) = (h^2 - r^2)\xi^i_\alpha y^\alpha \). Then
\[
w^i_\alpha(y) = -2\xi^a_\alpha \xi^i_\beta y^\beta + (h^2 - r^2)\xi^i_\beta \delta^\alpha_\beta
\]
and
\[
w^i_\beta(y) = -2\xi^i_\alpha (\delta^\alpha_\beta y^\alpha + y^\alpha \delta^\beta_\alpha + y^\beta \delta^\alpha_\beta)
\]
if \( r < h \). Let \( T_{\alpha\beta} = \delta_{\alpha\beta} \delta^{\alpha\beta} + \delta^\alpha_\alpha \delta^\beta_\beta + \delta^\alpha_\beta \delta^\beta_\alpha \) and note that \( \int_{r \leq h} (h^2 - r^2)y^\alpha y^\beta \, dy = Q \delta_{\alpha\beta} h^{r+4} \) for some positive constant \( Q \).
Hence
\[
\int_{r \leq h} w^i_\alpha(y) w^j_\beta(y) \, dy = -\int_{r \leq h} w^i_\alpha(y) w^j_\beta(y) \, dy
\]
\[
= 2 \xi^i_\gamma \int_{r \leq h} (h^2 - r^2) \xi^j_\gamma y^\gamma (\delta^\alpha_\beta y^\beta + y^\alpha \delta^\beta_\alpha + y^\beta \delta^\alpha_\beta) \, dy
\]
\[
= 2Q h^{r+4} \xi^i_\alpha \xi^j_\beta T_{\alpha\beta} \text{ since } w^i \text{ vanishes on } r = h.
\]
Let \( \xi^i_\alpha = w^i(y) \). Then \( \zeta \) is Lipschitzian with support \( \zeta \subset G \) and \( \zeta_n(x) = w^i(y) \lambda^a_\alpha \) if \( r < h \). Let \( \phi(p) = f(x_0, z(x_0), p) \) and \( a^\alpha_{\beta\gamma} = \phi^\beta_\gamma(p(x_0)) \). Then
\[
0 \leq a^\alpha_{\beta\gamma} \int_G \zeta(x) \zeta_n(\frac{x}{\lambda}) \, dx
\]
\[
= a^\alpha_{\beta\gamma} \lambda^\beta_\alpha \lambda^\gamma_\beta \int_{r \leq h} w^i(y) w^j_\beta(y) \, dy \text{ det } \lambda \, |^{-1} \, dy
\]
\[
= 2Q h^{r+4} \text{ det } \lambda \, |^{-1} a^\alpha_{\beta\gamma} \lambda^\beta_\alpha \lambda^\gamma_\beta T_{\alpha\beta}.
\]
Thus we have

**Theorem.** If $z$ is a weak relative minimum for $I$ then

(*)\[ a_{ij}^p \lambda_i^p \lambda_j^p \xi^i \lambda^j \geq 0 \]

for all $\lambda^i, \cdots, \lambda^j \in R_v$ and $\xi_v \in R^N$.

If we set $\lambda^i=\lambda$, $\xi^i=\xi$ and $\lambda^0=0$, $\xi^0=0$ for $p \neq 1, c \neq 1$, then we get the Legendre-Hadamard condition

(**)\[ a_{ij}^p \lambda^i \lambda^j \xi^i \lambda^j \geq 0 \]

for all $\lambda \in R_v$ and $\xi \in R^N$. If $f \in C^r$ then $f$ is quasi-convex if the Legendre-Hadamard condition holds [2, p. 112]. Let us say that $f$ is pseudo-convex if $f \in C^r$ and (*) holds. (In fact, Morrey defined quasi-convexity without imposing differentiability conditions on $f$.) Within the class of $C^r$ functions it is evident that pseudo-convexity implies quasi-convexity.

Let us say [2, p. 114] that $f$ is strongly quasi-convex if $f \in C^r$ and if

\[ \int_G f(x_0, z_0, p_0 + \zeta(x)) \, dx \geq f(x_0, z_0, p_0) \cdot m(G) \]

for any constant $(x_0, z_0, p_0)$, any bounded domain $G$, and any Lipschitz $\zeta$ with support $\zeta \subset \subset G$. Let $\zeta^i(x)=(p_0)\zeta^i$ so that $\zeta^i=p_0$. Let $F(x, z, p)=f(x_0, z_0, p)$ if $f$ is strongly quasi-convex and if $\{\zeta_n\}$ is a sequence of Lipschitz functions with support $\zeta_n \subset \subset G$ and $\|\zeta_n\| \to 0$, then $I_F(z) \leq \liminf_{n \to \infty} I_F(z + \zeta_n)$. Thus $F$, and hence $f$, satisfies (*) so that $f$ is pseudo-convex if $f$ is strongly quasi-convex. Thus to show that there exist quasi-convex functions which are not strongly quasi-convex it is sufficient to show that there exist functions satisfying (**) but not (*).

**References**


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