A NECESSARY CONDITION IN THE CALCULUS OF VARIATIONS

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Abstract. A necessary condition that an extremal be a relative minimum is derived. The condition includes and may be stronger than the Legendre-Hadamard condition.

By considering a slightly more general variation than Hadamard we will obtain a necessary condition in the calculus of variations which may be stronger than the Legendre-Hadamard condition [1, p. 253] and [2, p. 11].

If \( \phi \in L(E, F) \) and \( p \in E \) let \([p, \phi] = \phi(p)\) where \( E \) and \( F \) are vector spaces. If \( g \in L(E, L(E, F)) \) and if \( h, k \in E \) let \( ghk = [k, [h, g]] \) and \( gh^{(2)} = ghh \).

Let \( e_\alpha \in R^\alpha, \alpha = 1, \cdots, v, \) and \( E_i \in R^N, i = 1, \cdots, N, \) be defined by

\[
e_\alpha = (\delta^1_\alpha, \cdots, \delta^v_\alpha) \quad \text{and} \quad E_i = (\delta^1_i, \cdots, \delta^N_i).
\]

Let \( e^\phi \in L(R^\alpha, R) = R^\alpha \) be defined by \( e^\phi e_\alpha = \delta^\alpha_\phi \). Let \( e^\phi E_i \) be that element of \( L(R^\alpha, R^N) \) defined by \( (e^\phi E_i)e_\alpha = \delta^\alpha_i E_i \). If \( \lambda \in R_+ \) and \( \xi \in R^N \) then \( (\lambda \xi)e_\alpha = (\lambda e^\phi E_i)e_\alpha = \lambda \delta^\alpha_\phi E_i \) (tensor convention for summation). If \( \phi \in C''(L(R^\alpha, R^N), F) \) we write \( \phi^\lambda \) for \( \phi^\phi e^\phi E_i \) and \( \phi^{(2)} \) for \( \phi^\phi e^\phi E_i E_i \). Thus if \( \lambda = \lambda e^\phi \in R_+ \) and \( \xi = \xi_i E_i \in R^N \),

\[
\phi^{(2)}_i \lambda \lambda^{(2)} \xi^i \xi^i = \phi'' e^\phi E_i E_i, \lambda \lambda^{(2)} \xi^i \xi^i = \phi''(\lambda \xi)^{(2)}.
\]

Let \( G \) be a bounded domain in \( R^\alpha \) and \( f \in C''(G \times R^N \times L(R^\alpha, R^N), R) \). Let \( C' = C'(G, R^N) \) and if \( z \in C' \) let \( \|z\| = \sup \{ \|z(x)\| + \|z'(x)\| \mid x \in G \} \). If \( z \in C' \) let \( I_z(x) = \int_G f(x, z(x), z'(x)) \, dx \) where \( p = z' \). Suppose that \( I_z(z) \leq I_z(z + \xi) \) whenever \( \xi \in C' \), support \( \zeta \subset \subset G \) and \( \|z\| \) is small enough. Then \( \phi''(0) \geq 0 \) where \( \phi(\lambda) = I(z + \lambda \xi) \) whenever \( \xi \in C' \) and support \( \zeta \subset \subset G \). Thus

\[
\int_G \{ f_{zz}(P)\xi^{(2)}(x) + 2f_{zp}(P)\xi'(x) + f_{pp}(P)\xi^{(2)}(x) \} \, dx \geq 0
\]

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where $P=(x, z(x), p(x))$. Let $x_0 \in G$, let $n$ be so large that
\[ n^{-1}(-x_0 + \text{support } \zeta) \subset (-x_0 + G) \]
and let $\zeta_n$ be defined on $G$ with support $\zeta_n \subset x_0 + n^{-1}(-x_0 + \text{support } \zeta)$ by $\zeta_n(x_0 + y) = \zeta(x_0 + ny)n^{-1}$. Evidently support $\zeta_n \subset G$, $\zeta_n \in C'$, and
\[ \zeta_n(x_0 + y) = \zeta(x_0 + ny). \]
Using the continuity of $f^\nu$ and letting $n \to \infty$ we see that
\[ f_{pp}(x_0, z(x_0), p(x_0)) \int_G \zeta_n^{\nu}(x) \, dx \geq 0. \]

By approximations we obtain

**Lemma.** If $z$ is a weak relative minimum for $I$ then
\[ f_{pp}(x_0, z(x_0), p(x_0)) \int_G \zeta^{\nu}(x) \, dx \geq 0 \]
for all Lipschitzian $\zeta$ with support $\zeta \subset G$.

Now let $\xi_1, \ldots, \xi_n \in \mathbb{R}^N$ and $\lambda_1, \ldots, \lambda_n$ be a linearly independent set in $\mathbb{R}^n$. Let $y^s = \lambda^s_2(x^0 - x_0^0)$ and $r=\|y\|$. Let $h>0$ be so small that $x \in G$ if $r \leq h$. If $i=1, \ldots, N$ let $w^i$ be defined on $G' = \{y| x \in G\}$ with support $w^i$ contained in $r \leq h$ by $w^i(y) = (h^2 - r^2)\xi^i y^s$. Then
\[ w^i(y) = -2y^s \xi^i y^s + (h^2 - r^2)\xi^i \delta^s_\nu \]
and $w^i(y) = -2\xi^i (\delta_\nu y^s + y^s \delta_\nu) + y^s \delta^i_\nu$ if $r < h$. Let $T^i = \delta_\nu \delta^i + \delta^i_\nu \delta_\nu + \delta^i_\nu \delta^i$, and note that $\int_{r \leq h} (h^2 - r^2)y^s dy = Q\delta_\nu h^{r+4}$ for some positive constant $Q$. Hence
\[ \int_{r \leq h} w^i(y)w^j(y) \, dy = -\int_{r \leq h} w^i(y)w^j(y) \, dy \]
\[ = 2\xi^j \int_{r \leq h} (h^2 - r^2)y^s (\delta_\nu y^s + y^s \delta_\nu) \, dy \]
\[ = 2Qh^{r+4} \xi^j \xi^i T^i \quad \text{since } w^j \text{ vanishes on } r = h. \]

Let $\xi^i(y) = w^i(y)$. Then $\xi$ is Lipschitzian with support $\zeta \subset G$ and $\xi^i(x) = w^i(y)\lambda^s_\nu$ if $r < h$. Let $\phi(p) = f(x_0, z(x_0), p)$ and $\alpha^s_\nu = \phi^s_\nu(p(x_0))$. Then
\[ 0 \leq \alpha^s_\nu \int_G \xi^i(x) \xi^j(x) \, dx \]
\[ = \alpha^s_\nu \lambda^s_\nu \lambda^s_\nu \int_{r \leq h} w^i(y)w^j(y) \, dy \]
\[ = 2Qh^{r+4} \int \det \lambda^{-1} \alpha^s_\nu \lambda^s_\nu \lambda^s_\nu \lambda^s_\nu T^s_\nu. \]
Thus we have

**Theorem.** If \( z \) is a weak relative minimum for \( I \) then

\[
(*) \quad a_{ij}^s \lambda_i^s \lambda_j^s \xi^i T_{ij}^{ps} \geq 0
\]

for all \( \lambda, \cdots, \lambda^s \in R_v \) and \( \xi_1, \cdots, \xi_v \in R^N \).

If we set \( \lambda^s = \lambda, \xi_1 = \xi \) and \( \lambda^p = 0, \xi_o = 0 \) for \( p \neq 1, o \neq 1 \), then we get the Legendre-Hadamard condition

\[
(**) \quad a_{ij}^s \lambda \xi^i T_{ij}^{ps} \geq 0
\]

for all \( \lambda \in R_v \) and \( \xi \in R^N \). If \( f \in C^r \) then \( f \) is quasi-convex if the Legendre-Hadamard condition holds [2, p. 112]. Let us say that \( f \) is pseudo-convex if \( f \in C^r \) and \( (*) \) holds. (In fact, Morrey defined quasi-convexity without imposing differentiability conditions on \( f \).) Within the class of \( C^r \) functions it is evident that pseudo-convexity implies quasi-convexity.

Let us say [2, p. 114] that \( f \) is strongly quasi-convex if \( f \in C^r \) and if

\[
\int_G f(x_0, z_0, p_0 + \zeta'(x)) \, dx \geq f(x_0, z_0, p_0) \cdot m(G)
\]

for any constant \( (x_0, z_0, p_0) \), any bounded domain \( G \), and any Lipschitz \( \zeta \) with support \( \zeta \subset \subset G \). Let \( z'(x) = (p_0)_{\zeta} \xi_d \) so that \( z' = p_0 \). Let \( F(x, z, p) = f(x_0, z_0, p) \). if \( f \) is strongly quasi-convex and if \( \{\zeta_n\} \) is a sequence of Lipschitz functions with support \( \zeta_n \subset \subset G \) and \( \|\zeta_n\| \rightarrow 0 \), then \( I_F(z) \leq \lim \inf_{n \rightarrow \infty} I_F(z + \zeta_n) \). Thus \( F \), and hence \( f \), satisfies \( (*) \) so that \( f \) is pseudo-convex if \( f \) is strongly quasi-convex. Thus to show that there exist quasi-convex functions which are not strongly quasi-convex it is sufficient to show that there exist functions satisfying \( (**) \) but not \( (*) \).

**References**


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