ON DIRECT PRODUCTS OF REGULAR \( p \)-GROUPS

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Abstract. We prove that, for each prime \( p \), there exists a regular \( p \)-group \( H(p) \) with the property that, if \( G \) is a regular \( p \)-group and \( G \times H(p) \) is regular, then the derived group of \( G \) has exponent \( p \). This provides a strong converse to a theorem of Grün.

Introduction. It has long been known that the direct product of regular \( p \)-groups need not be regular. The first example was due to H. Wielandt and may be found in [2, III, 10.3]. In the positive direction, however, Grün [3] has shown that if \( G \) is a regular \( p \)-group whose derived group has exponent \( p \), then \( G \times H \) is regular for every regular \( p \)-group \( H \). The purpose of this note is to prove a strong converse to this theorem. We will prove:

Theorem. For each prime \( p \), there exists a regular \( p \)-group \( H(p) \) with the following property:

If \( G \) is a regular \( p \)-group and \( G \times H(p) \) is also regular, then the derived group of \( G \) has exponent dividing \( p \).

We recall that a finite \( p \)-group \( G \) is regular, if, whenever \( g, h \in G \), there exists an element \( d \) of the derived group of the group generated by \( g \) and \( h \) such that \( (gh)^p = g^ph^pd^p \). If, in addition, \( G' \) has exponent \( p \), then \( (gh)^p = g^ph^p \) for all \( g, h \) belonging to \( G \) and so \( G \) is \( p \)-abelian in the sense of Baer [1]. Conversely, if \( G \) is \( p \)-abelian, then \( G \) is evidently regular and it follows from standard results on regular groups (see, for example, [2, III, 10]), that \( G' \) has exponent dividing \( p \). Hence, recalling the quoted theorem of Grün, we have

Corollary. Let \( G \) be a regular \( p \)-group. Then \( G \times H \) is regular, for each regular \( p \)-group \( H \), if and only if \( G \) is \( p \)-abelian.

As regular \( 2 \)-groups are abelian, the Theorem is trivial if \( p = 2 \) and so we will henceforth assume that \( p \) is an odd prime. We denote the commutator \( g^{-1}h^{-1}gh \) of elements \( g \) and \( h \) of a group \( G \) by \( [g, h] \)—and similarly for higher commutators—and the derived group of \( G \) by \( G' \). Also, \( (g, h) \) will denote an element of the direct product \( G \times H \), where \( g \in G \) and \( h \in H \). All groups considered in this note are finite.

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Proof of the Theorem. We begin by constructing the group $H(p)$. We have some freedom of choice in this; in fact it will suffice that our group have the following properties:

(i) $H(p)$ can be generated by two elements $a$ and $b$.

(ii) $H(p)$ is regular but not $p$-abelian.

(iii) Every commutator, in $H(p)$, of weight 3 or more has order dividing $p$.

(iv) $(ab)^p = a^p b^p$ and $[a, b]^p \neq 1$.

Our construction is essentially due to Paul M. Weichsel [4], who proves a similar theorem under the restriction that $G$ be metabelian. We repeat it here largely for convenience (and because it involves a slight twist on that construction), but refer to [4] for fuller details. Let $A$ denote the direct product $(a_1) \times (a_2) \times \cdots \times (a_p)$, where $(a_i)$ is a cyclic group, of order $p^2$ if $i=1$ or 2 and of order $p^i$ if $i \geq 3$. Let $t$ denote $(p-1)/2$ and let $b$ denote the automorphism of $A$ defined by:

\[
\begin{align*}
 b: a_i & \rightarrow a_i a_{i+1} \quad (1 \leq i < p-1),
 b: a_{p-1} & \rightarrow a_{p-1} a_{2t}.
\end{align*}
\]

Then the group, $H(p)$, that we require, is the split extension of $A$ by $\langle b \rangle$. It can be verified, either by direct calculation or by reference to [4], that $b$ is, in fact, an automorphism of $A$ and that the group $H(p)$ satisfies the required conditions, with $a_1 = a$. In particular, property (iv) is verified as follows:

\[
(ab)^p = a^p b^p [a, b]^t = a^p b^p
\]

since $[a, b, \cdots, b] = [a, b]^p$.

Let $G$ be a $p$-group which is regular but not $p$-abelian; we will show that $G \times H(p)$ is irregular. It suffices to assume that every proper subgroup and every proper homomorphic image of $G$ is $p$-abelian—for, otherwise, we could take a section of $G$ with these properties. We will now extract a few relevant properties of $G$.

As $G$ is not $p$-abelian, there exist elements $g$ and $h$ of $G$ such that $(gh)^p \neq g^p h^p$ and, by the minimality of $G$, these must generate $G$. Let $M$ be a central subgroup of order $p$. By the minimality of $G$, $G/M$ is $p$-abelian and so $G'/M$ has exponent $p$. Thus, if $g_1, h_1$ and $k$ are arbitrary elements of $G$, $[g_1, h_1]^p \in M$ and so $[[g_1, h_1], k] = 1$. Hence, by a standard result on regular groups [2, III, 10.6], $[g_1, h_1, k]^p = 1$ and so every commutator of weight 3 or more has order dividing $p$. Thus, as $G$ is regular, $(gh)^p = g^p h^p$, for some integer $r$. But $[g, h]^p \in M$ and so, as $(gh)^p \neq g^p h^p$, $[g, h]$ has order precisely $p$ and $r$ is prime to $p$.

The proof of the theorem now follows very quickly. For, suppose that $G \times H(p)$ were regular, and let $x = (g, a)$ and $y = (h, b)$. As $G$ and $H(p)$
both have the property that commutators of weight 3 or more have order dividing $p$, $G \times H(p)$ also has this property. Hence $(xy)^p = x^py^p[x, y]^{ps}$ for some integer $s$. But,

$$(xy)^p = (gh, ab)^p = ((gh)^p, (ab)^p)$$

$$= (g^ph^p[g, h]^{pr}, a^pb^p) = x^py^p([g, h]^{pr}, 1).$$

Thus,

$$([g, h]^{pr}, 1) = [x, y]^{ps} = ([g, h]^{ps}, [a, b]^{ps}),$$

and so $[g, h]^{pr} = [g, h]^{ps}$ and $[a, b]^{ps} = 1$. But $[a, b]^{pr} \neq 1$ and therefore $p | s$. It follows that $[g, h]^{pr} = 1$—a contradiction which completes the proof of the theorem.

REFERENCES


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