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RESTRICTIONS OF FOURIER TRANSFORMS OF CONTINUOUS MEASURES

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ABSTRACT. Let G denote a compact abelian group and Γ its discrete dual. It is proved that $E\subset\Gamma$ is Sidon if and only if the restriction to E of the algebra of Fourier transforms of continuous measures on G is all of $l_\infty(E)$.

The set E is said to be a Sidon set if there is a constant $c>0$ such that for every E-polynomial $P(x)=\sum a_j(\gamma_j, x)$, $\gamma_j\in E$,

$$\|P\|_\infty \geqq c\sum |a_j|.$$

Several characterizations of the Sidon property are known (cf. [**4**, p. 121]). Here the most important is the following:

E is Sidon if and only if the restrictions to E of the algebra of Fourier transforms of measures on G, $M(G)^\wedge$, is $l_\infty(E)$, the space of bounded complex functions on E.

Our characterization of Sidon sets relies heavily on Drury's recent work on Sidon sets. If μ denotes a measure, we write μ_c and μ_d for its continuous and discrete parts respectively. The Fourier-Stieltjes transform of μ will be denoted by $\hat{\mu}$.

Recall that the space of weakly almost periodic functions on Γ, WAP(Γ), taken in the uniform norm is a Banach space. It consists of those continuous functions on Γ whose set of translates is relatively weakly compact as a subset of the bounded continuous functions on Γ. For $f\in$ WAP(Γ) the mean of f will be denoted by $\mathscr{M}(f)$. It is the unique constant function lying in the closed convex hull of the set of translates of f. It is important to recall here that $M(G)^\wedge\subset$ WAP(Γ). Finally we denote by AP(Γ) the space of almost periodic functions on Γ.

LEMMA (GLICKSBERG). *Suppose $E\subset\Gamma$ is such that no finite union of translates of E contains Γ. Then $\Gamma\backslash E$ determines mean values, i.e. $f\in$ WAP(Γ), $f\geqq 0$ on $\Gamma\backslash E$ implies $\mathscr{M}(f)\geqq 0$. In particular if $f, g\in$ WAP(Γ) and $f=g$ on $\Gamma\backslash E$ then $\mathscr{M}(f-g)=0$.*

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PROOF. Assume that $f \in \mathrm{WAP}(\Gamma)$ and $f \geqq 0$ on $\Gamma \backslash E$. Since $\mathscr{M}(f)$ lies in the closure of the convex hull of the translates of f we may choose $c_i \geqq 0$, $\sum c_i = 1$ and $\gamma_i \in \Gamma$, $i=1, 2, \cdots, n$, such that

$$\left| \sum_i c_i f(\gamma - \gamma_i) - \mathscr{M}(f) \right| < \varepsilon \quad \text{for all } \gamma \in \Gamma.$$

Since $\Gamma \not\subset \bigcup_{i=1}^n (\gamma_i + E)$ we may find a $\gamma \in \Gamma \backslash \bigcup_{i=1}^n (\gamma_i + E)$ so that $f(\gamma - \gamma_i) \geqq 0$, $i=1, 2, \cdots, n$. Thus $\mathscr{M}(f) \geqq -\varepsilon$ for all $\varepsilon > 0$, so $\mathscr{M}(f) \geqq 0$, and the proof is complete.

If Γ is infinite and $E \subset \Gamma$ is Sidon then E satisfies the hypothesis of the lemma, because by Drury's theorem **[1]** any finite union of translates of E must again be Sidon. However, Γ itself can be Sidon only in case it is finite.

THEOREM. *The set E is Sidon if and only if the algebra $M_c(G)^\wedge$ restricted to E is all of $l_\infty(E)$.*

PROOF. Suppose that E is a Sidon set. First we need to recall a result due to Eberlein **[2]**. There is a projection of norm 1 from $\mathrm{WAP}(\Gamma)$ into $\mathrm{AP}(\Gamma)$. For a measure μ on G we have the decomposition $\hat{\mu}(\gamma) = \hat{\mu}_d(\gamma) + \hat{\mu}_c(\gamma)$, for all $\gamma \in \Gamma$, with $|\hat{\mu}_c(\gamma)|$ having mean square zero and $\hat{\mu}_d$ almost periodic. Thus Eberlein's theorem says $\|\hat{\mu}\|_\infty \geqq \|\hat{\mu}_d\|_\infty$.

Eberlein's theorem may in fact be extended as follows: If $\Gamma \backslash E$ determines mean values, then there is a projection of norm 1 from $\mathrm{WAP}(\Gamma \backslash E)$ onto $\mathrm{AP}(\Gamma \backslash E)$, where by $\mathrm{WAP}(\Gamma \backslash E)$ (resp. $\mathrm{AP}(\Gamma \backslash E)$) we mean $\mathrm{WAP}(\Gamma) | \Gamma \backslash E$ (resp. $\mathrm{AP}(\Gamma) | \Gamma \backslash E$), and both spaces are taken in the uniform norm on $\Gamma \backslash E$.

Now a function in $\mathrm{WAP}(\Gamma)$ may be regarded as a continuous function on the weak almost periodic compactification of Γ. If E is any subset of Γ and $f \in \mathrm{WAP}(\Gamma \backslash E)$, then by Tietze's extension theorem f is the restriction of an F (not necessarily unique) in $\mathrm{WAP}(\Gamma)$ having the same norm.

To see the generalization of Eberlein's theorem mentioned above, suppose that $\Gamma \backslash E$ determines mean values and that $f \in \mathrm{WAP}(\Gamma \backslash E)$. Let F be any norm preserving extension of f to Γ. By Eberlein's theorem $F = f_1 + f_2$ where f_1 is almost periodic, $|f_2|$ has mean square zero and $\|f_1\|_\infty \leqq \|F\|_\infty$. Now if F' is any other extension of f to a weakly almost periodic function on Γ we have as before $F' = f_1' + f_2'$, f_1' almost periodic, $|f_2'|$ of mean square zero. Since $\Gamma \backslash E$ determines mean values we see that $|f_1 - f_1'|$ has mean square zero, and thus $f_1 = f_1'$. Thus the mapping $f \to f_1 | \Gamma \backslash E$ is well defined and norm decreasing. It only remains to check that it is linear. Suppose $g \in \mathrm{WAP}(\Gamma \backslash E)$, $G \in \mathrm{WAP}(\Gamma)$ is a Tietze extension of g to Γ, and g_1 is its almost periodic part. Set $h = f + g$, and let H be a Tietze extension for h to Γ and h_1 its almost periodic part. Since $F + G$ and H agree on

$\Gamma\backslash E$, $|f_1+g_1-h_1|$ has mean square zero, and thus $f_1+g_1=h_1$ proving the linearity.

The result of [1] now gives a $\mu \in M(G)$ such that

$$\hat{\mu}(\gamma) = 1, \quad \gamma \in E,$$
$$|\hat{\mu}(\gamma)| < \varepsilon, \quad \text{otherwise.}$$

From the extension of Eberlein's theorem given above we see that

$$\sup_{\gamma\in\Gamma\backslash E} |\hat{\mu}_d(\gamma)| \leqq \sup_{\gamma\in\Gamma/E} |\hat{\mu}(\gamma)|$$

and

$$\sup_{\gamma\in\Gamma\backslash E} |\hat{\mu}_d(\gamma)| = \|\hat{\mu}_d\|.$$

The latter is so since $\hat{\mu}_d$, restricted to $\Gamma\backslash E$, has a unique norm preserving extension to an almost periodic function on Γ, namely, to $\hat{\mu}_d$.

Thus we have

$$|\hat{\mu}_c(\gamma) - 1| \leqq \varepsilon, \quad \gamma \in E,$$
$$|\hat{\mu}_c(\gamma)| \leqq 2\varepsilon, \quad \text{otherwise.}$$

Since E is Sidon we have a $\nu \in M(G)$ such that $\hat{\nu}(\gamma)=1/\hat{\mu}_c(\gamma)$, $\gamma \in E$. Hence $\lambda=\nu * \mu_c$ is a continuous measure such that $\hat{\lambda}=1$ on E. Now if $\xi(\gamma)$ is an arbitrary bounded function on E and $\sigma \in M(G)$ is such that $\hat{\sigma}(\gamma)=\xi(\gamma)$ for $\gamma \in E$, then $\sigma * \lambda$ is a continuous measure enjoying the same property.

REMARK. Professor Irving Glicksberg has pointed out that via the lemma another proof of the theorem may be given using an argument similar to that of Theorem 1 of [3] to show $\hat{\mu}_d(\Gamma)\subset\hat{\mu}(\Gamma\backslash E)^-$ where $^-$ denotes closure.

Since preparation of this manuscript the author has learned that the theorem of this note was discovered independently by S. Hartman. Finally, the author would like to thank Professor I. Glicksberg for several helpful comments.

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