A PROBLEM IN ADDITIVE NUMBER THEORY
DONALD QUIRING

Abstract. For every real number \( \alpha, 0 < \alpha < 1 \), a sequence \( A = \{a_1, a_2, \ldots \} \) is constructed for which the density of \( A \) is \( \alpha \) and \( A \) has the following property: Given any \( n \) distinct positive integers \( \{b_1, b_2, \ldots, b_n\} \) the sequence consisting of all numbers of the form \( a_i + b_j \) has density \( 1 - (1 - \alpha)^n \).

Let \( A = \{a_1, a_2, \ldots\} \) and \( B = \{b_1, b_2, \ldots\} \) be increasing sequences of positive integers. The sequence \( A + B \) is defined as the increasing sequence consisting of all the sums \( a_i + b_j \). Let \( A(n) \) be the number of elements of \( A \) that are less than \( n \). The limit \( A(n)/n \), if it exists, is called the density of \( A \) and designated \( d(A) \).

P. Erdös and A. Renyi [1] have shown that for every \( \alpha, 0 < \alpha < 1 \), there exists a sequence \( A \) of density \( \alpha \) which has the property that for any infinite sequence \( B \), \( d(A + \{b_1, \ldots, b_n\}) = 1 - (1 - \alpha)^n \). This implies \( d(A + B) = 1 \). The purpose of this paper is to provide examples of such sequences.

If \( \alpha \) is rational we proceed as follows. Express \( \alpha \) as a quotient of natural numbers, \( \alpha = p/q, q > p \). List all the natural numbers in order in base \( q \) notation to obtain a sequence

\[ S = \{s_1, s_2, \ldots\}, \quad 0 \leq s_i \leq q - 1. \]

Define \( A \) by \( A = \{i \mid 0 \leq s_i \leq p - 1\} \). Then \( d(A) = \alpha \) and if \( B \) is an increasing sequence of positive integers \( d(A + \{b_1, \ldots, b_n\}) = 1 - (1 - \alpha)^n \).

We prove that \( A \) has these properties in the case \( \alpha = 1/2 \). The other cases can be handled by essentially the same method.

List the natural numbers in order in base 2 notation separated by hyphens as follows:

\[ 1-10-11-100-101-110-111-\ldots. \]

We treat this list as a sequence of digits \( s_1, s_2, \ldots \) with hyphens between \( s_1 \) and \( s_2 \), \( s_3 \) and \( s_4 \), etc. Define \( \{(s_i, t_i, u_i)\} \) by letting \( s_i = 0 \) or \( 1 \) be the \( i \)th entry in the above sequence \( t_i = \inf \), (there is one hyphen between the \( i \)th entry and the \( i+j \)th entry), and \( u_i = \inf \) (there is one hyphen between the \( i \)th entry and the \( i+j \)th entry).

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Let $A = \{i|s_i = 0\}$. We show that $d(A + \{b_1, \cdots, b_n\}) = 1 - 2^{-n}$ for an arbitrary increasing sequence $B$. The case $n=1$ will then give us $d(A) = 1/2$.

Define sequences $T_n$ and $T_n^k$ by

$$T_n = \{i | t_i \geq b_n + 2\} \quad \text{and} \quad T_n^k = \{i | t_i \geq b_n + 2, u_i = k\}.$$ 

Let $i_m = \inf_i (u_i = m + 1)$; that is,

$$i_m = 1 + \sum_{k=1}^{m} k2^{k-1} = (m - 1)2^m + 2.$$ 

Define a sequence $C_n$ as the intersection of $T_n$ with the complement of $A + \{b_1, \cdots, b_n\}$. Since $d(T_n)$ is clearly equal to 1 it suffices to show $d(C_n) = 2^{-n}$. Note that $C_n = \{i| i \in T_n \}$ and $s_{i-b_n} = s_{i-b_n+1} = \cdots = s_{i-b_n} = 1$ and that for $i_m \leq i < i_{m+1}$, $T_n(i) = \sum_{k=1}^{m-b_n} kT_n^k(i)$. $T_n(i)$ is the number of elements of $T_n$ that are less than $i$.

Among any $2^{b_n+k}$ consecutive elements of $T_n^k$ there are $2^{b_n+k-n}$ elements of $C_n$. This is because among any $2^{b_n+k}$ consecutive natural numbers every possible combination of the last $b_n+k$ digits appears exactly once.

Therefore, for all $i$,

$$(C_n \cap T_n^k)(i) - 2^{b_n+k-n} \leq 2^{-n}T_n^k(i) \leq (C_n \cap T_n^k)(i) + 2^{b_n+k-n},$$

and for $i_m \leq i < i_{m+1}$, $m > b_n$,

$$C_n(i) - 2^{b_n+k-n} \leq 2^{-n}T_n(i) \leq C_n(i) + 2^{b_n+k-n}.$$ 

So $C_n(i) - 2^{m-n} < 2^{-n}T_n(i) < C_n(i) + 2^{m-n}$ and $|C_n(i)/i - 2^{-n}T_n(i)/i| < 2^{m-n}/i$. Since we are assuming $i \geq i_m > (m-1)2^m$, we have $|C_n(i)/i - 2^{-n}T_n(i)/i| < 2^{-n}/(m-1)$. We know that $\lim_{i \to \infty} T_n(i)/i = d(T_n) = 1$ and it follows that $\lim_{i \to \infty} |C_n(i)/i - 2^{-n}| \leq 2^{-n}/(m-1)$ for all $m$. We conclude that $d(C_n) = 2^{-n}$ and the proof is complete.

We turn now to the case where $\alpha$ is not rational. Express $\alpha$ as a limit of rational numbers $\alpha_j$, $\alpha = \lim \alpha_j$, and let $A_{\alpha_j}$ be the sequence with density $\alpha_j$ constructed above. Compose $A_\alpha$ of increasingly long segments of the sequences $A_{\alpha_j}$ as follows.

Define inductively integers $N_j$ and finite sequences $E_j$ by

$$E_j = (A_{\alpha_1} \cap (1, N_1)) \cap (A_{\alpha_2} \cap (N_1 + 1, N_2)) \cap \cdots \cap (A_{\alpha_j} \cap (N_{j-1} + 1, N_j)),$$

choosing $N_j$ large enough that

$$\sup C(E_j + C)(N_j/N_j - (1 - (1 - \alpha_j)^n) < 1/j$$

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where the supremum is taken over all subsets \( C \) of \( \{1, 2, \ldots, j\} \) and \( n \) is the number of elements of \( C \). If we let \( A_a = \bigcup E_j \) then \( A_a \) clearly has the desired property.

Reference


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NEW MEXICO 87106

Current address: Department of Mathematics, Louisiana State University, New Orleans, Louisiana 70122