THE SOLUTION OF A NONLINEAR
GRONWALL INEQUALITY

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Abstract. This paper extends some of the earlier results of J.
V. Herod, W. W. Schmaedeke and G. R. Sell, and B. W. Helton
and shows that, under the given conditions,
(1) there is a function \( u \) satisfying the inequality
\[
  f(x) \leq h(x) + (RL) \int_a^x (fG + fH)
\]
such that, if \( f \) satisfies the given inequality, then \( f(x)^u(x) \); and
(2) there is a function \( u \) satisfying the inequality
\[
  0 < f(x) \leq k + (RL) \int_a^x [(fG_1 + f^{p_1}G_2) + (fH_1 + f^{p_2}H_2)],
\]
where \( n \) is a positive integer and \( p = \pm 1 \) and \( pn \neq 1 \), such that, if \( f \)
satisfies the given inequality, then \( f(x)^u(x) \).

Definitions and notations. \( R \) is the set of real numbers; capital
letters and lower case letters denote functions from \( R \times R \) to \( R \) and from
\( R \) to \( R \), respectively; \( f^{-1} = 1/f \); all integrals are subdivision-refinement type
limits; \( (LR) \int_a^x (fG + gH) \sim \int f(x_{i-1})G(x_{i-1}, x_i) + g(x_i)H(x_{i-1}, x_i); \) \( G \in OA^o \)
on \([a, b]\) if \( \int_a^b G \) exists and \( \int_a^b |G - f G| = 0 \); \( G \in OM^o \) on \([a, b]\) if \( \int_a^b (1+G) \)
exists for \( a \leq x < y \leq b \); \( G \in OB^o \) on \([a, b]\) if \( \int_a^b (1+G) \)
exists for \( a \leq x < y \leq b \). For detailed definitions see \([1]\) and \([2]\).

Lemma 1. If \( G \in OA^o \) and \( OB^o \) on \([a, b]\), then \( G \in OL^o \) on \([a, b]\).

Lemma 2. If \( \int_a^b G^2 = 0 \), \( \int_a^b H^2 = 0 \) and \( f \) is a bounded function, then
\( \int_a^b GH = 0 \) and \( \int_a^b fG^2 = 0 \).

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equations.
**Lemma 3.** If $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $H \in OL^\circ$ and $G \in OA^\circ$ and $OB^\circ$ on $[a, b]$ then $HG \in OA^\circ$ and $OM^\circ$ on $[a, b]$ [2, Theorem 2].

**Lemma 4.** If $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ and $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that on $[a, b]$ $f$ has bounded variation, $H$ and $G \in OA^\circ$ and $OB^\circ$, and $(1-G)^{-1}$ exists and is bounded, then on $[a, b]$, $(1-G)^{-1} \in OL^\circ$, and each of $df(1-G)^{-1}, H(1-G)^{-1}, f(x)H(x,y), f(y)H(x,y)$ and $(1+H) \cdot (1-G)^{-1} - 1 = (H+G)(1-G)^{-1}$ belongs to $OA^\circ$ and $OM^\circ$.

Lemma 4 is a corollary to Lemma 3.

**Lemma 5.** If $f$ is quasicontinuous and $G \in OA^\circ$ and $OB^\circ$ on $[a, b]$, then $fG \in OA^\circ$ and $OB^\circ$ on $[a, b]$.

Lemma 5 is a corollary to Lemma 3.

**Lemma 6.** Given, $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, $c \in \mathbb{R}$ and $c > 0$; $H$ and $G \in OA^\circ$ and $OB^\circ$, $H \geq 0$, $G \geq 0$ and $1 - G \geq c$ on $[a, b]$; and $u$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that $u$ is bounded above on $[a, b]$,

$$(LR) \int_a^b (uH + uG) \exists \text{ and } u(x) \subseteq (LR) \int_a^b (uH + uG) \text{ for } x \in [a, b].$$

Conclusion. If $x \in [a, b]$, then $u(x) \subseteq 0$ [2, Theorem 3].

**Lemma 7.** Given, $f$ and $h$ are functions from $\mathbb{R}$ to $\mathbb{R}$ and $H$, $G$ and $B$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $f(a) = h(a)$, $h$ has bounded variation, $(1-G)^{-1}$ exists and is bounded, $dh(1-G)^{-1} \in OA^\circ$, $B = (1+H)(1-G)^{-1}$ and $B - 1 \in OB^\circ$ and $OM^\circ$ on $[a, b]$.

Conclusion. The following statements are equivalent:

1. $(LR) \int_a^b (fH + fG)$ exists and $f(x) = h(x) + (LR) \int_a^b (fH + fG)$ for $x \in [a, b]$; and
2. if $a < x \leq b$, then $(L) \int_a^x |f(i)[B - \int B]| = 0$ and $f(x) = f(a) \int_a^x B + (R) \int_a^x dh(1 - G)^{-1} i_i B$.

Lemma 7 is a special case of Theorem 5.1 [1, p. 310].

**Lemma 8.** If $G$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and $G \in OB^\circ$, then the following statements are equivalent: (1) $G \in OA^\circ$ and (2) $G \in OM^\circ$ [1, Theorem 3.4].

**Lemma 9.** If $[a, b]$ is a number interval and $F$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $\int_a^b |FG| = 0$, $\int_a^b G^2 = 0$, and $F$ and $G \in OB^\circ$ and $OM^\circ$ on $[a, b]$, then

1. $\int_a^b (1+F) \int_a^b (1+G) = \int_a^b (1+F+G)$;
2. if $n$ is a positive integer, then $\int_a^b (1+G)^n = \int_a^b (1+nG)$; and
3. if $\int_a^b (1-G)$ exists, then $\int_a^b (1+G)^{-1} = \int_a^b (1-G).$
Lemma 9 is a special case of Theorem 5.6 [1, p. 315].

**Lemma 10.** If $G$ is a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $\int_a^b G$ exists, then $G \in O\Lambda^0$ on $[a, b]$ [1, Theorem 4.1].

**Theorem 1.** Given $c \in \mathbb{R}$ and $c > 0$; $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ such that $H$ and $G \in O\Lambda^0$ and $O\Lambda^0$, $H \geq 0$, $G \geq 0$ and $1 - G \geq c$ on $[a, b]$; $f$ and $h$ are functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f$ is bounded and $h$ has bounded variation on $[a, b]$, $(LR) \int_a^b (fH + fG)$ exists and, if $x \in [a, b]$, then

$$f(x) \leq h(x) + (LR)\int_a^x (fH + fG).$$

**Conclusion.** (1) If $B = (1 + H)(1 - G)^{-1}$, then the function

$$u(x) = h(a)\int_a^x B + (R)\int_a^x dh(1 - G)^{-1} \int_a^x B$$

exists on $[a, b]$ and satisfies the inequality in the hypothesis.

(2) If $x \in [a, b]$, then $f(x) \leq u(x)$.

**Proof.** If $a \leq x < y \leq b$, then $\int_a^y B$ exists and $B - 1 \in O\Lambda^0$ (Lemma 4) and $B - 1 \in O\Lambda^0$. Since $G \in O\Lambda^0$ and $O\Lambda^0$ and $1 - G \geq c > 0$, then $G$, $1 - G$ and $(1 - G)^{-1} \in O\Lambda^0$; hence, $dh(1 - G)^{-1} \in O\Lambda^0$ on $[a, x]$ (by Lemma 3). Since $B - 1 \in O\Lambda^0$, it follows that, for each $x \in [a, b]$, the function, $\int_a^x B$ has bounded variation and is quasicontinuous on $[a, x]$; since $dh(1 - G)^{-1} \in O\Lambda^0$ and $O\Lambda^0$ on $[a, x]$, it follows from Lemma 5 that $dh(1 - G)^{-1} \int_a^x B \in O\Lambda^0$ on $[a, x]$. Let $u$ be the function such that, if $x \in [a, b]$, then

$$u(x) = h(a)\int_a^x B + (R)\int_a^x dh(1 - G)^{-1} \int_a^x B.$$

It follows from Lemma 7 that, if $x \in [a, b]$, then

$$u(x) = h(x) + (LR)\int_a^x (uH + uG)$$

and $f(x) - u(x) \leq (LR)\int_a^x [(f - u)H + (f - u)G]$ and, from Lemma 6, that $f(x) - u(x) \leq 0$.

Theorem 4 in [2] is a special case of the above theorem. If $f$ has bounded variation, the above inequality is a special case of the one given by Herod [3, Remark, p. 36]; however, Theorem 1 gives the best upper bound of the solution set for this inequality and this type of integral.

**Theorem 2.** Given (1) $[a, b]$ is a number interval, $n$ is a positive integer, $k > 0$, $p = 1$ or $p = -1$, and $p + 1 \neq 1$; (2) $G_1$, $G_2$, $H_1$, $H_2$ are functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and each belongs to $O\Lambda^0$ and $O\Lambda^0$, $G_1 \geq 0$, $H_1 \geq 0$, $pG_2 \geq 0$ and $pH_2 \geq 0$ on $[a, b]$ and $\int_a^b G_1^2 = \int_a^b G_2^2 = \int_a^b H_1^2 = \int_a^b H_2^2 = 0$; and (3) $f$ is a bounded...
function from $R$ to $R$ such that $f$ is bounded away from zero on $[a, b]$ and, if $x \in [a, b]$, then \( (L) \int_a^x f^m G_m \) and \( (R) \int_a^x f^m H_m \) exist for $m = 1, 2, \cdots, n$, and the following integral exists, and
\[
0 < f(x) \leq k + (LR) \int_a^x \left[ (f G_1 + f^p G_2) + (f H_1 + f^p H_2) \right].
\]

**Conclusion.** (1) If $x \in [a, b]$, then $g(x) = \frac{1}{(1 - pn)} \left( \frac{1 - G_1 + H_1}{1 + \beta_1 G_1 + \beta_2 H_1} \right)$ and \( (LR) \int_a^x \left( g^{p_n-1} G_2 + g^{p_n-1} H_2 \right) \) exist. (2) If $u$ and $h$ are functions from $R$ to $R$ such that, on $[a, b]$,
\[
h(x) = k^{1-pn} + (1 - pn) (LR) \int_a^x \left( g^{p_n-1} G_2 + g^{p_n-1} H_2 \right) > 0
\]
and
\[
u(x) = g(x) h(x) \frac{1}{(1 - \beta_n)},
\]
then $f(x) \leq u(x)$ for $x \in [a, b]$. Furthermore, on $[a, b]$ the function $u$ is bounded away from zero and satisfies the inequality stated in the hypothesis.

**Proof.** Since $G_1$ and $H_1 \in OA^0$ and $OB^0$ on $[a, b]$, then $G_1 + H_1 \in OA^0$ and $OB^0$ and, by Lemmas 1 and 8, $G_1 + H_1 \in OM^0$ and $OL^0$ and $-(G_1 + H_1) \in OB^0$, $OA^0$, $OL^0$ and $OM^0$ on $[a, b]$. Hence, $g(x) = \frac{1}{(1 - pn)} \left( 1 + G_1 + H_1 \right)$ exists for $x \in [a, b]$ and, by Lemma 9, \( g^{-1}(x) = \frac{1}{a} \left[ 1 - (G_1 + H_1) \right] \) for $x \in [a, b]$. If $a \leq x < y \leq b$, then there is a number $e$ such that
\[
g(y) - g(x) = \frac{1}{a} \left[ 1 + G_1 + H_1 \right] \left[ x \right] \left( 1 + G_1 + H_1 \right) - 1
\]
\[
\frac{1}{a} \left[ 1 + G_1 + H_1 \right] \left( 1 + G_1 + H_1 \right) + e = \frac{1}{a} \left[ 1 + G_1 + H_1 \right] \left( 1 + G_1 + H_1 \right) + e.
\]
Therefore, since $G_1 + H_1 \in OM^0$ and since \( \int_a^y G_2 = 0 \) and \( \int_a^y H_2 = 0 \), then $g$ is continuous on $[a, b]$ and, since $G_1 + H_1 \in OB^0$, then $g$ has bounded variation on $[a, b]$. Similarly, $g^{-1}$ is continuous and has bounded variation on $[a, b]$. Since $G_2$ and $H_2 \in OA^0$ and $OB^0$ and since $g$ and $g^{-1}$ are continuous on $[a, b]$, then \( (LR) \int_a^x \left( g^{p_n-1} G_2 + g^{p_n-1} H_2 \right) \) exists (Lemma 5). Since \( \int_a^y G_2 = \int_a^y H_2 = 0 \), then $h$ is continuous on $[a, b]$.

Let $u$ be the function such that $u(x) = g(x) h(x) \frac{1}{(1 - \beta_n)}$ on $[a, b]$; then $u$ and $u^{-1}$ exist and are continuous and, if $x \in [a, b]$, then
\[
u(x)^{1-pn} \frac{g(x)^{p_n-1}}{g(x)^{p_n-1}} = h(x)
\]
\[
= k^{1-pn} + (LR) \int_a^x \left\{ (u^{1-pn} g^{p_n-1}) \left[ u^{p_n-1}(1 - pn)G_2 \right] + u^{1-pn} \left[ u^{p_n-1}(1 - pn)H_2 \right] \right\}
\]
\[
= k^{1-pn} (LR) \int_a^x \left\{ 1 + \left[ u^{p_n-1}(1 - pn)G_2 \right] \right\} \cdot \left\{ 1 - \left[ u^{p_n-1}(1 - pn)H_2 \right] \right\}^{-1},
\]
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by Lemma 7. Hence, if $x \in [a, b]$, then

$$u(x) = g(x)k(LR)\int_a^x (1 + u^{p-1}G_2)\left(1 - u^{p-1}H_2\right)^{-1}$$

and

$$g(x) = \int_a^x (1 + G_1 + H_1) = \int_a^x (1 + G_1)\int_a^x (1 + H_1)$$

(Lemmas 8, 9)

and

$$u(x) = k(LR)\int_a^x [1 + (G_1 + u^{p-1}G_2)][1 - (H_1 + u^{p-1}H_2)]^{-1}$$

(Lemma 9).

Therefore, if $x \in [a, b]$, then

$$u(x) = k + (LR)\int_a^x [(uG_1 + u^{p-1}G_2) + (uH_1 + u^{p-1}H_2)].$$

Hence,

$$f(x) - u(x) \leq (LR)\int_a^x [(f - u)G_1 + (f^{p-1} - u^{p-1})G_2]$$

$$+ [(f - u)H_1 + (f^{p-1} - u^{p-1})H_2]$$

$$= (LR)\int_a^x [(f - u)(G_1 + vG_2) + (f - u)(H_1 + vH_2)],$$

where

$$v = p(f^{n-1} + f^{n-2}u + \cdots + u^{n-1})(f^{-n}u^{-(1-p)/2}).$$

Since $f, f^{-1}, u$ and $u^{-1}$ are bounded on $[a, b]$, then $v$ is bounded on $[a, b]$ and, since $\int_a^b H_1^2 = \int_a^b H_2^2 = 0$, then it follows from Lemma 2 that $\int_a^b (H_1 + vH_2)^2 = 0$ and hence $1 - (H_1 + vH_2)^2 = 1$. It follows from the hypothesis that $(R)\int_a^b f^rH_2$ exists and $f^rH_2 \in OB^o$ for $r = 0, 1, 2, \cdots, n-1$ and that $f^rH_2 \in OA^o$ (Lemma 10). Since $u$ and $u^{-1}$ are continuous, then

$$(R)\int_a^b f^{n-r-1}(f^{-n}u^{-(1-p)/2})H_2$$

exists for $r = 1, 2, \cdots, n-1$ and $p = \pm 1$ (Lemma 5). Therefore, $(R)\int_a^b vH_2$ and $(R)\int_a^b (H_1 + vH_2)$ exist and $H_1 + vH_2 \in OA^o$ (Lemma 10). Similarly, $G_1 + vG_2 \in OA^o$. Since $f \geq 0$ and $u \geq 0$ on $[a, b]$, then $pv \geq 0$ and it follows from the hypothesis that $G_1 + vG_2$ and $H_1 + vH_2$ are nonnegative on $[a, b]$. It follows from Lemma 6 that $f - u \leq 0$ on $[a, b]$.

**Note 1.** If $p = 1$, then the requirement that $f$ be bounded away from zero may be deleted.

**Note 2.** If $pn = 0$ or $pn = 1$, then Theorem 2 can be simplified to a special case of Theorem 1.
Note 3. If $f^{xn}$ is quasicontinuous, then the integrals in part (3) of the hypothesis will exist.

**Example.** Suppose that $f$ and $h$ are functions from $R$ to $R$ such that, on $[a, b]$, $h$ is continuous, $h \geq 0$, $f(a) \geq 1$, $f$ is bounded and bounded away from zero, $f'$ exists and is continuous, and $f'(x) \leq [f(x) - f_2(x)]h(x)$; then

$$f(x) \leq f(a) + \int_a^x [fh \, dt - f_2h \, dt].$$

The hypothesis of Theorem 2 is satisfied with $G_1 = h \, dt$, $G_2 = -h \, dt$, $H_1 = H_2 = 0$, $n = 2$ and $p = -1$. Hence, if $x \in [a, b]$, then

$$g(x) = \int_a^x (1 + h \, dt) = \exp \int_a^x h \, dt,$$

and

$$f(x) \leq u(x) = \left( \exp \int_a^x h \, dt \right) \left( f(a)^3 + 3(L) \int_a^x \left( \exp \int_a^t h \, dq \right)^{-3} (-h \, dt) \right)^{1/3}$$

$$= \left( \exp 3 \int_a^x h \, dt \right) [f(a)^3 - 1] + 1 \right)^{1/3}.$$

If $0 \leq a < b$ and $f$ and $h$ are functions such that $f(x) = e^{x(1+e^{-3x})^{1/3}}$ and $h(x) = 1 + x$ for $x \in [a, b]$, then $f$ and $h$ satisfy the hypothesis of the above example.

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