MULTIVALUED NONEXPANSIVE MAPPINGS
AND OPIAL'S CONDITION

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Abstract. We give relations between a condition introduced by Z. Opial which characterizes weak limits by means of the norm in some Banach spaces and approximations of the identity, in particular for systems of projections. Finally a fixed point theorem for multivalued nonexpansive mappings in a Banach space satisfying this condition is proved; this result generalizes those of J. Markin and F. Browder.

1. Introduction. Let $X$ be a Banach space, $C$ a convex weakly compact subset of $X$ and $T$ a nonexpansive mapping from $C$ into $C$, i.e.
\[ \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C. \]

Related to the problem of existence of a fixed point for $T$ and its approximation, Z. Opial [9] introduced an inequality for weak convergent sequences characterizing its limits; we take this property as

Definition 1.1. Let $X$ be a Banach space. $X$ satisfies Opial's condition if for each $x$ in $X$ and each sequence $\{x_n\}$ weakly convergent to $x$
\[ \lim \inf \|x_n - y\| > \lim \inf \|x_n - x\| \]
holds for $y \neq x$.

An equivalent definition [2] is obtained replacing (1.1) by
\[ \lim \sup \|x_n - y\| > \lim \sup \|x_n - x\|, \]
this following from
\[ \lim \inf \|x_n - y\| = \lim_{k} \|x_{nk} - y\| > \lim \sup \|x_{nk} - x\| \geq \lim \inf \|x_n - x\| \]
and from
\[ \lim \sup \|x_n - x\| = \lim_{j} \|x_{nj} - x\| < \lim \inf \|x_{nj} - y\| \leq \lim \sup \|x_n - y\| \]
for convenient subsequences of $\{x_n\}$.

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These results are part of the author's doctoral thesis (University of Brussels, 1970).
Hilbert spaces and $l^p$ spaces ($1 < p < \infty$) satisfy Opial's condition, also finite dimensional Banach spaces.

The existence of a fixed point in the problem mentioned in the beginning of this paragraph is known when $X$ satisfies Opial's condition, but from results of [5] this is a corollary of a theorem of W. Kirk [6], so this condition becomes interesting in the approximation of a fixed point for a "univalued" nonexpansive mapping (see [9]).

A useful simplification of (1.1) is given in the following

**Lemma 1.1.** A Banach space $X$ satisfies Opial's condition if and only if

\begin{equation}
\lim \inf \|x_n\| = 1 \Rightarrow \lim \inf \|x_n - x\| > 1
\end{equation}

for $x \neq 0$, where $\rightarrow$ denotes weak convergence of the sequence \{\(x_n\)\}.

**Proof.** Let $y_n \rightarrow y$; if $\lim \inf \|y_n - y\| = 0$, then (1.1) follows from the unicity of a weak limit. If not take $a = \lim \inf \|y_n - y\|, a > 0$, and $x_n = a^{-1}(y_n - y)$; then $x_n \rightarrow 0$ and $\lim \inf \|x_n\| = 1$, so (1.3) gives

\[
\lim \inf \|x_n - x\| > 1
\]

for $x \neq 0$. Replacing $x_n$ we obtain

\[
\lim \inf \|y_n - y - ax\| > \lim \inf \|y_n - y\|
\]

for $x \neq 0$, i.e. for any $y - ax \neq y$. The inverse implication is obvious. Q.E.D.

2. Approximation of the identity. We consider a directed family of continuous linear operators of finite rank which approach the identity of a Banach space $X$, i.e. a family \{\(P_j; j \in J, P_j\) linear and continuous\}, where $J$ has an "increasing" order denoted by $\infty$ such that

\[
P_j : X \rightarrow X_j, \quad \dim X_j < \infty, \quad X_j \text{ subspace of } X,
\]

\[
\lim P_j x = x, \quad \text{for each } x \in X.
\]

We shall call such a family an approximation of the identity. For example when a Banach space possesses a Schauder basis, the associated system of projections constitutes an approximation of the identity.

**Theorem 2.1.** Let $X$ be a Banach space and let \{\(P_j; j \in J\)\} be an approximation of the identity. If for each $c > 0$ there exists $r = r(c) > 0$ such that

\begin{equation}
\|x - P_j x\| = 1 \quad \text{and} \quad \|P_j x\| \geq c \Rightarrow \|x\| \leq 1 + r
\end{equation}

holds for $j \succ j_0, j_0$ fixed in $J$, then $X$ satisfies Opial's condition.

**Proof.** We apply Lemma 1.1 and verify (1.3), so we take $x_n \rightarrow 0$ with $\lim \inf \|x_n\| = 1$ and $x \neq 0$. If we define $Q_j = I - P_j$, where $I$ is the identity on
\( X, \) we have
\[ \| x_n \| - (\| P_j x_n \| + \| Q_j x \|) \leq \| Q_j (x_n - x) \| \leq \| x_n - x \| + \| P_j (x_n - x) \| \]
and
\[ \| P_j x \| \leq \| P_j (x_n - x) \| + \| P_j x_n \|. \]
Then
\[ 1 - \| Q_j x \| \leq \liminf_n \| Q_j (x_n - x) \| \leq \limsup_n \| Q_j (x_n - x) \| \]
\[ \leq \limsup_n \| x_n - x \| + \| P_j x \| , \]
\[ \| P_j x \| \leq \liminf_n \| P_j (x_n - x) \| , \]
\[ (2.3) \]
since \( \dim X < \infty \) and \( P_j \) is linear and continuous.
From (2.1) we have \( \lim_j Q_j x = 0 \) and \( \lim_j P_j x = x, x \neq 0 \); then in (2.3)
\[ 0 < c' \leq \liminf_n \| Q_j (x_n - x) \| \leq \limsup_n \| Q_j (x_n - x) \| \leq M , \]
\[ 0 < c'' \leq \liminf_n \| P_j (x_n - x) \| \]
holds for \( j \gg j_1 (x) \) with \( c', c'' \) and \( M \) independent of \( j \). Hence there exists \( n_0(j) \) such that
\[ c'/2 \leq \| Q_j (x_n - x) \| \leq 2M \quad \text{and} \quad 0 < c''/2 \leq \| P_j (x_n - x) \| \]
holds for \( j \gg j_1 (x) \) and \( n \geq n_0 (j) \). For the same \( j \) and \( n \),
\[ (2.4) \]
\[ \frac{Q_j (x_n - x)}{\| Q_j (x_n - x) \|} = 1 \quad \text{and} \quad \frac{P_j (x_n - x)}{\| Q_j (x_n - x) \|} \geq c > 0 \]
with \( c \) independent of \( j \) and \( n \).
For \( j \gg j_0, j \gg j_1 (x), n \geq n_0 (j) \) and from (2.2) and (2.4) we deduce
\[ (2.5) \]
\[ \| x_n - x \| \geq (1 + r) \| Q_j (x_n - x) \|. \]
Hence from the first inequality in (2.3) and from (2.5)
\[ (2.6) \]
\[ \liminf_n \| x_n - x \| \geq (1 + r) (1 - \| Q_j x \|) \]
As we can choose \( j \) large enough such that \((1 + r)(1 - \| Q_j x \|) > 1\), we finally obtain \( \liminf \| x_n - x \| > 1 \). Q.E.D.

All known examples of Banach spaces satisfying Opial's condition are isomorphic to uniformly convex Banach spaces; due to this the following two corollaries become interesting.

**COROLLARY 2.1.** There exists a reflexive Banach space not isomorphic to any uniformly convex Banach space which satisfies Opial's condition.
Proof. Let us put $X_n = \mathbb{R}^n$ endowed with the norm $\|x_n\|_n = (\sum_{i=1}^{n-1} |x_n(i)|^n)^{1/n}$, $x_n = (x_n(1), \cdots, x_n(n)) \in \mathbb{R}^n$. The Hilbert product $X = \{x = (x_n)_{n=1,2,\ldots}; x_n \in X_n \text{ and } \sum_{n=1}^{\infty} \|x_n\|^2_n < \infty\}$ is known to verify the first part of Corollary 2.1 (see [3]) for the norm $\|x\| = (\sum_{n=1}^{\infty} \|x_n\|^2_n)^{1/2}$.

Condition (2.2) is easily proved for the projections

$$(P_n x)_k = x_k, \quad k \leq n,$$

$$= 0, \quad k > n,$$

and $\{P_n; n=1, 2, \cdots\}$ forms an approximation of the identity. Q.E.D.

**Corollary 2.2.** There exist nonreflexive Banach spaces which satisfy Opial’s condition.

Proof. The usual Schauder basis of $l^1$ with its set of associated projections verify the hypothesis of Theorem 2.1. Q.E.D.

If we restrict a Banach space to be uniformly convex, the family can be asked to fulfill a more simple condition and we obtain the same conclusion.

**Theorem 2.2.** Let $X$ be a uniformly convex Banach space with an approximation of the identity $\{P_j; j \in J\}$ such that

$$\lim_j \|I - P_j\| = 1,$$

where $I$ is the identity on $X$. Then $X$ satisfies Opial’s condition.

Proof. Let $x_n \to 0$ with $\liminf \|x_n\| = 1$. From (2.7) we have for $x \in X$ and $Q_j = I - P_j$ that

$$\|x_n - x\| \geq \|Q_j(x_n - x)\| - \varepsilon \geq \|x_n\| - (\|P_j x_n\| + \|Q_j x\| + \varepsilon),$$

where $\varepsilon = \varepsilon(j)$ is independent of $n$ and $\lim_j \varepsilon(j) = 0$. Then

$$\liminf \|x_n - x\| \geq 1 - (\|Q_j x\| + \varepsilon),$$

and taking limits on $j$ we obtain

$$\liminf \|x_n - x\| \geq 1$$

for every $x$ in $X$. If for some $x \neq 0$ the equality holds in (2.8) then, $X$ being uniformly convex, we would have

$$\liminf \|x_n - ax\| < 1,$$

contradicting (2.8), so the inequality is strict for $x \neq 0$ and from Lemma 1.1 we deduce the desired result. Q.E.D.
A consequence of this theorem for $L^p$ spaces is the following

**Corollary 2.3.** Every Schauder basis $(e_k)_{k=1,2,...}$ in $L^p(A)$, where $A$ is the unit real interval with Lebesgue measure and $1 < p < \infty$, $p \neq 2$, is such that

\[
\liminf_n \| I - P_n \| > 1
\]

for $P_n(\sum_{k=1}^\infty a_k e_k) = \sum_{k=1}^n a_k e_k$ and $I$ the identity.

**Proof.** $L^p(A)$ is uniformly convex and does not satisfy Opial’s condition (see [9]), so (2.9) follows from Theorem 2.2. Q.E.D.

**Remark 1.** The product $\ell^2 \times \ell^2$ endowed with the norm $\|(a, y)\| = \max(|a|, \|y\|)$ does not satisfy Opial’s condition, even though $\ell^2$ satisfy it. This can be shown by taking $x_n = (0, e_n)$, for $e_n$ the usual $n$th element of the basis of $\ell^2$. Then $\|x_n\| = 1$ and $x_n \to 0$, but for $x = (1, 0)$ we have $\|x_n - x\| = 1$.

**Remark 2.** Let us recall that a Banach space possesses normal structure if every convex and bounded subset $A$ of $X$, $A$ with diameter $\delta(A) > 0$, has a point $x \in A$ such that $\sup_{y \in A} \|x - y\| < \delta(A)$. A similar condition to (2.2) was used in [4] for obtaining normal structure of a Banach space. Explicitly, for every $c > 0$ there exists $r = r(c) > 0$ such that

\[
\|P_j x\| = 1 \quad \text{and} \quad \|x - P_j x\| \geq c \Rightarrow \|x\| \geq 1 + r,
\]

where the family $\{P_j\}$ is an approximation of the identity. (2.10) does not imply Opial’s condition as the example in Remark 1 shows it; however in [5] it is proved that Opial’s condition implies normal structure.

3. **Nonexpansive multivalued mappings.** We apply Opial’s condition to obtain a fixed point for a nonexpansive compact-valued mapping.

**Definition 3.1.** Let $C$ be a nonempty convex weakly compact subset of a Banach space $X$. A mapping $T : C \to K(X)$, where $K(X)$ denotes the family of nonempty compact subsets of $X$, is nonexpansive if

\[
D(Tx, Ty) \leq \|x - y\|,
\]

for $x, y \in C$ and $D(\cdot, \cdot)$ the Hausdorff metrics on $K(X)$.

If we recall that the graph $G(U)$ of a multivalued mapping $U : A \to 2^Y$ is

\[
G(U) = \{(x, y) \in X \times Y; x \in A, y \in Ux\}
\]

we can prove the

**Theorem 3.1.** Let $T : C \to K(X)$ be nonexpansive and let $X$ satisfy Opial’s condition. Then the graph of $U = I - T$ is closed in $X$, $\sigma(X, X^*) \times (X, \|\cdot\|)$, where $I$ denotes the identity on $X$, $\sigma(X, X^*)$ the weak topology and $\|\cdot\|$ the norm (or strong) topology.
Proof. As the domain of $U$ is weakly compact we must prove that the graph is only sequentially closed. Let $(x_n, y_n) \in G(U)$ be such that

\[(3.2) \quad x_n \to x, \quad y_n \to y.\]

We must see that $x \in C$ and $y \in Ux = x - Tx$. That $x \in C$ is clear. As $y_n \in x_n - Tx_n$ we can write

\[(3.3) \quad y_n = x_n - v_n, \quad v_n \in Tx_n.\]

From (3.1) we can find $v'_n \in Tx$ such that

\[(3.4) \quad \|v_n - v'_n\| \leq \|x_n - x\|;\]

this is an easy consequence of the definition of Hausdorff metrics.

From (3.3) and (3.4) we obtain passing to limits on $n$

\[(3.5) \quad \lim \inf \|x_n - x\| \geq \lim \inf \|v_n - v'_n\| \geq \lim \inf \|x_n - y_n - v'_n\|.
\]

$Tx$ being compact and $y_n \to y$, for a convenient subsequence still denoted $v'_n$ we have $v'_n \to v \in Tx$, so from (3.5)

\[(3.6) \quad \lim \inf \|x_n - x\| \geq \lim \inf \|x_n - y - v\|.
\]

As $x_n \to x$, Opial’s condition implies that $y + v = x$, so $y = x - v \in x - Tx = Ux$. Q.E.D.

Let us recall the following known generalization of Picard’s theorem:

**Proposition 3.1** [7]. If $T: C \to K(C)$ is contractive, i.e. $D(Tx, Ty) \leq r\|x - y\|$ for $x, y \in C$ and $0 < r < 1$, then there exists a fixed point $x_0 \in Tx_0$.

We are now able to prove our fixed point theorem.

**Theorem 3.2.** Let $X$ be a Banach space which satisfies Opial’s condition. If $C$ is a nonempty convex weakly compact subset of $X$ and $T: C \to K(C)$ is a compact-valued nonexpansive mapping, then there exists a fixed point $x_0 \in Tx_0$.

Proof. Let $x' \in C$ be fixed and define for $0 < r_m < 1$ and $r_m \to 1$

\[(3.7) \quad T_m x = r_m Tx + (1 - r_m)x',\]

in an obvious way. Then $T_m: C \to K(C)$ and $T_m$ is contractive, so from Proposition 3.1 there exists a fixed point $x_m \in T_m x_m$. $C$ being weakly compact, for a convenient subsequence $\{x_n\}$ of $\{x_m\}$ we have $x_n \to x_0 \in C$.

From (3.7) we deduce

\[x_n = r_n v_n + (1 - r_n)x', \quad v_n \in Tx_n, \quad x' \in C,\]

so

\[\|x_n - v_n\| = (1 - r_n)\|x' - v_n\|.\]
Then $y_n = x_n - v_n \in (I-T)x_n$ and $y_n \to 0$. If we put $U = I-T$, we have that $(x_n, y_n) \in G(U)$ and

\begin{equation}
 x_n \to x_0, \quad y_n \to 0.
\end{equation}

From Theorem 3.1 we obtain

\[ 0 \in Ux_0 = x_0 - Tx_0, \]

i.e. $x_0 \in Tx_0$ is a fixed point for $T$. Q.E.D.

**Remark.** Recently Theorem 3.1 has been applied in [1] to obtain a generalization of Theorem 3.2 which only requires that $T$ be nonexpansive and sends the boundary of $C$ into compact subsets of $C$, this result can also be obtained from Theorem 3.1 and results on the topological degree for contractive multivalued mappings due to R. Nussbaum [8].

If we restrict the Banach space $X$ to be a Hilbert space we obtain a theorem due to J. Markin [7], and if we consider Banach spaces with a weakly continuous duality mapping we obtain a result of F. Browder [2], because these spaces satisfy Opial’s condition [5].

**Bibliography**


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