ON COMPOSITION SERIES IN FINITE GROUPS

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Abstract. Theorem. Let $G$ be a finite group and $H$ a solvable subgroup of $G$. Suppose that the Schreier conjecture holds. Then $G$ is solvable iff $G$ has an $H$-composition series.

Let $G$ be a group and $H \leq G$. Let $\{G_i\}_{i=0}^n$ be subnormal series with $G_0 = \{1\}$ and $G_n = G$. This series is called an $H$-composition series if $H$ normalizes each $G_i$, and if there exists no subgroup $X$ properly between $G_{i+1}$ and $G_i$ which is normalized by $H$.

If $G$ is a finite solvable group then for all $H \leq G$ such $H$-composition series exist. These can be obtained by refinement into irreducible $H$-factors of any chief series of $G$. If $G$ is not solvable then, for particular $H$, such series may not exist. This is easily seen by letting $G$ be simple nonabelian and $H$ any proper subgroup.

The object of this note will be to shed some light on restrictions that one must have on finite groups $G$ and $H \leq G$ if such $H$-composition series occur. All groups are finite. If $\{G_i\}_{i=0}^n$ is a subnormal series of $G$ we denote by $G(i)$ the factor $G_{i-1}/G_i$ and call $\{G(i)\}_{i=0}^n$ the factors of the series. A factor of $G$ is a group $R/S$ where $S \leq R \leq G$. If $K/L$ is a factor of $G$ then we can in a natural way define $\text{Aut}_G(K/L)$ as $N(K) \cap N(L)/C(K/L)$ and $\text{Out}_G(K/L)$ as $N(K) \cap N(L)/KC(K/L)$. These groups correspond to the automorphisms and outer automorphisms that $G$ induces on the factor $K/L$. If $\Sigma$ is a group, then $\Sigma$ is said to be involved in $G$ if $\Sigma$ is isomorphic to some factor of $G$. If $\Sigma$ is a nonabelian simple group, then $K/L$ is called a $\Sigma$-factor if it is the direct product of isomorphic copies of $\Sigma$.

If $\Sigma$ is a nonabelian simple group the Schreier conjecture states that $\text{Out}(\Sigma) = \text{Aut}(\Sigma)/\text{In}(\Sigma)$ is a solvable group. In what follows, if $K/L$ is a simple nonabelian factor of $G$ then if $\text{Out}_G(K/L)$ is solvable we will say that $G$ satisfies the Schreier conjecture with respect to the factor $K/L$. Our result is

Theorem. Let $H \leq G$ with $H$-composition series $\{G_i\}_{i=0}^n$. Let $\Sigma$ be a nonabelian simple group and $G(i)$ be a $\Sigma$-factor. Suppose $G$ satisfies the Schreier conjecture with respect to the factor $K/L$. Our result is
conjecture with respect to the simple summands of $G^{(i)}$. Then $\Sigma$ is involved in $H$.

(Note that the simple summands of $G^{(i)}$ are all conjugate by elements of $H$ and thus induced automorphism groups are isomorphic.)

**Lemma 1.** Let $G$ be a semidirect product of $K$ by $H$. If $H$ is maximal in $G$ and solvable then $G$ is solvable.

**Proof.** By induction on $|G|$ we may assume that $\text{core}_G(H) = 1$. Let $R/K$ be minimal normal in $G$. We have that $R/K$ is a $p$ group and $H = N(R \cap H)$. It follows that $R \cap H \in \text{Syl}_p(R)$ since if not we get $R \cap H < N_R(R \cap H)$ which together with $R \cap H < H$ implies that $R \cap H < G$. Let $S \in \text{Syl}_p(K)$. The Frattini argument gives that $G = K \cdot N(S)$. Since $(|K|, p) = 1$ we get, by Sylow's theorem and a suitable choice of $S$, that $R \cap H \leq N_R(S)$. The Frattini argument applied to $R \cap N \leq N_R(S) < N(S)$ yields that $N(S) = N_H(S) \cdot N_R(S)$. Since $R = K \cdot (R \cap H)$, it follows by Dedekind's theorem that $N_R(S) = K \cdot (R \cap H) \cdot N(S) = (R \cap H) \cdot N_R(S)$. Thus we have that $N(S) = N_H(S) \cdot N_R(S)$ or that $G = N_H(S) \cdot K$. Since $G = HK$, $H \cap K = 1$, we arrive at $N_H(S) = H$ or $H < N(S)$. This forces $K = S$ and thus $G$ is solvable.

**Lemma 2.** Let $G$ be a semidirect product of $K$ by $H$ with $H$ maximal in $G$. Suppose $K$ is a $\Sigma$-factor where $\Sigma$ is a nonabelian simple group. If $G$ satisfies the Schreier conjecture for any simple direct summand of $K$ then $\Sigma$ is involved in $H$.

**Proof.** Let $S$ be a simple direct summand of $K$. Then $S$ is isomorphic to $\Sigma$. We can choose $h_1, \ldots, h_t$ a full set of coset representatives of $N_H(S)$ in $H$ and $K = S^{h_1} \times \cdots \times S^{h_t}$. Suppose a $1 < R \leq S$ such that $N_H(S)$ normalizes $R$. Since for $x \in N_H(S), \exists 1 \leq l \leq k, y \in N_H(S)$, such that $h_l x h_l^{-1} = y \cdot h_l$ we get that $R^{h_1} \times \cdots \times R^{h_t}$ is normalized by $H$. This yields that $R=S$. Now induction applies to the semidirect product of $S$ by $N_H(S)$. If $|S \cdot N_H(S)| < |G|$ we conclude that $\Sigma$ is involved in $N_H(S)$ and therefore in $H$. Thus we can conclude that $K = S$. Let $T = C(S)$. Then $T \lhd G$ and $T \cap S = 1$. If $T \not\lhd H$ since $H$ is maximal we get that $G = HT$. It follows that $S \not\leq ST/T \leq ST \cap H/T \cap H$ and again $\Sigma$ is involved in $H$. If $T \leq H$ we look at $G/T$. Our assumption of the Schreier conjecture yields $G/ST$ and thus $H/T$ solvable. Thus Lemma 1 applies to make $G/T$ solvable. This final contradiction, since $ST/T$ is not solvable, proves Lemma 2.

The proof of our theorem follows easily from Lemma 2. By the definition of $H$-composition series it is easy to see that $H$ either covers or avoids each $G^{(i)}$. If $H$ covers this factor then surely $G^{(i)}$ and thus $\Sigma$ is involved in $H$. 

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If $H$ avoids $G^{(i)}$ then we are in the situation that $HG_{i-1}/G_i$ is a semidirect product of $G_{i-1}/G_i$ by $HG_i/G_i$. By the $H$-irreducibility of $G^{(i)}$ we have that $HG_i/G_i$ is maximal in $HG_{i-1}/G_i$. By our Lemma 2 we are done. Note that $HG_{i-1}/G_i$ satisfies the Schreier conjecture with respect to any simple summand of $G^{(i)}$.

**Corollary.** Let $H \leq G$ with $H$ solvable. Suppose that $\text{Out}_G(\Sigma)$ is solvable for all nonabelian simple factors $\Sigma$ of $G$. Then $G$ is solvable if and only if $G$ has an $H$-composition series.

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