ON COMPOSITION SERIES IN FINITE GROUPS

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Abstract. Theorem. Let \( G \) be a finite group and \( H \) a solvable subgroup of \( G \). Suppose that the Schreier conjecture holds. Then \( G \) is solvable iff \( G \) has an \( H \)-composition series.

Let \( G \) be a group and \( H \leq G \). Let \( \{G_i\}_0^n \) be subnormal series with \( G_n = \{1\} \) and \( G_0 = G \). This series is called an \( H \)-composition series if \( H \) normalizes each \( G_i \), and if there exists no subgroup \( X \) properly between \( G_{i+1} \) and \( G_i \) which is normalized by \( H \).

If \( G \) is a finite solvable group then for all \( H \leq G \) such \( H \)-composition series exist. These can be obtained by refinement into irreducible \( H \)-factors of any chief series of \( G \). If \( G \) is not solvable then, for particular \( H \), such series may not exist. This is easily seen by letting \( G \) be simple nonabelian and \( H \) any proper subgroup.

The object of this note will be to shed some light on restrictions that one must have on finite groups \( G \) and \( H \leq G \) if such \( H \)-composition series occur.

All groups are finite. If \( \{G_i\}_0^n \) is a subnormal series of \( G \) we denote by \( G^{(i)} \) the factor \( G_{i-1}/G_i \) and call \( \{G^{(i)}\}_1^n \) the factors of the series. A factor of \( G \) is a group \( R/S \) where \( S \trianglelefteq R \leq G \). If \( K/L \) is a factor of \( G \) then we can in a natural way define \( \text{Aut}_G(K/L) \) as \( N(K) \cap N(L)/C(K/L) \) and \( \text{Out}_G(K/L) \) as \( N(K) \cap N(L)/KC(K/L) \). These groups correspond to the automorphisms and outer automorphisms that \( G \) induces on the factor \( K/L \). If \( \Sigma \) is a group, then \( \Sigma \) is said to be involved in \( G \) if \( \Sigma \) is isomorphic to some factor of \( G \).

If \( \Sigma \) is a nonabelian simple group, then \( K/L \) is called a \( \Sigma \)-factor if it is the direct product of isomorphic copies of \( \Sigma \).

If \( \Sigma \) is a nonabelian simple group the Schreier conjecture states that \( \text{Out}(\Sigma) = \text{Aut}(\Sigma)/\text{In}(\Sigma) \) is a solvable group. In what follows, if \( K/L \) is a simple nonabelian factor of \( G \) then if \( \text{Out}_G(K/L) \) is solvable we will say that \( G \) satisfies the Schreier conjecture with respect to the factor \( K/L \). Our result is

Theorem. Let \( H \leq G \) with \( H \)-composition series \( \{G_i\}_0^n \). Let \( \Sigma \) be a nonabelian simple group and \( G^{(i)} \) be a \( \Sigma \)-factor. Suppose \( G \) satisfies the Schreier conjecture with respect to the factor \( K/L \). Our result is

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conjecture with respect to the simple summands of $G^{(i)}$. Then $\Sigma$ is involved in $H$.

(Note that the simple summands of $G^{(i)}$ are all conjugate by elements of $H$ and thus induced automorphism groups are isomorphic.)

**Lemma 1.** Let $G$ be a semidirect product of $K$ by $H$. If $H$ is maximal in $G$ and solvable then $G$ is solvable.

**Proof.** By induction on $|G|$ we may assume that $\text{core}_G(H) = 1$. Let $R/K$ be minimal normal in $G/K$. We have that $R/K$ is a $p$ group and $H = N(R \cap H)$. It follows that $R \cap H \in \text{Syl}_p(R)$ since if not we get $R \cap H < N(R)(R \cap H)$ which together with $R \cap H < H$ implies that $R \cap H < G$. Let $S \in \text{Syl}_p(K)$. The Frattini argument gives that $G = K \cdot N(S)$. Since $([K], p) = 1$ we get, by Sylow's theorem and a suitable choice of $S$, that $R \cap H \leq N_R(S)$. The Frattini argument applied to $R \cap N \leq N_R(S) < N(S)$ yields that $N(S) = N_H(S) \cdot N_R(S)$. Since $R = K \cdot (R \cap H)$, it follows by Dedekind's theorem that $N_R(S) = K \cdot (R \cap H) \cdot N(S) = (R \cap H) \cdot N_R(S)$. Thus we have that $N(S) = N_H(S) \cdot N_R(S)$ or that $G = N_H(S) \cdot K$. Since $G = HK$, $H \cap K = 1$, we arrive at $N_H(S) = H$ or $H < N(S)$. This forces $K = S$ and thus $G$ is solvable.

**Lemma 2.** Let $G$ be a semidirect product of $K$ by $H$ with $H$ maximal in $G$. Suppose $K$ is a $\Sigma$-factor where $\Sigma$ is a nonabelian simple group. If $G$ satisfies the Schreier conjecture for any simple direct summand of $K$ then $\Sigma$ is involved in $H$.

**Proof.** Let $S$ be a simple direct summand of $K$. Then $S$ is isomorphic to $\Sigma$. We can choose $h_1, \cdots, h_t$ a full set of coset representatives of $N_H(S)$ in $H$ and $K = S^{h_1} \times \cdots \times S^{h_t}$. Suppose a $1 < R \leq S$ such that $N_H(S)$ normalizes $R$. Since for $x \in N_H(S)$, $\exists 1 \leq l \leq k$, $y \in N_H(S)$, such that $h_i \times h_l = y \times h_i$ we get that $R^{h_1} \times \cdots \times R^{h_t}$ is normalized by $H$. This yields that $R = S$. Now induction applies to the semidirect product of $S$ by $N_H(S)$. If $|S \cdot N_H(S)| < |G|$ we conclude that $\Sigma$ is involved in $N_H(S)$ and therefore in $H$. Thus we can conclude that $K = S$. Let $T = C(S)$. Then $T < G$ and $T \cap S = 1$. If $T < H$ since $H$ is maximal we get that $G = HT$. It follows that $S \triangleleft ST/T \cong ST \cap H/T \cap H$ and again $\Sigma$ is involved in $H$. If $T \leq H$ we look at $G/T$. Our assumption of the Schreier conjecture yields $G/ST$ and thus $H/T$ solvable. Thus Lemma 1 applies to make $G/T$ solvable. This final contradiction, since $ST/T$ is not solvable, proves Lemma 2.

The proof of our theorem follows easily from Lemma 2. By the definition of $H$-composition series it is easy to see that $H$ either covers or avoids each $G^{(i)}$. If $H$ covers this factor then surely $G^{(i)}$ and thus $\Sigma$ is involved in $H$. 

If $H$ avoids $G^{(i)}$ then we are in the situation that $HG_{i-1}/G_i$ is a semidirect product of $G_{i-1}/G_i$ by $HG_i/G_i$. By the $H$-irreducibility of $G^{(i)}$ we have that $HG_i/G_i$ is maximal in $HG_{i-1}/G_i$. By our Lemma 2 we are done. Note that $HG_{i-1}/G_i$ satisfies the Schreier conjecture with respect to any simple summand of $G^{(i)}$.

**Corollary.** Let $H \leq G$ with $H$ solvable. Suppose that $Out_G(\Sigma)$ is solvable for all nonabelian simple factors $\Sigma$ of $G$. Then $G$ is solvable if and only if $G$ has an $H$-composition series.