SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON THE INVERSE FUNCTION THEOREM: A COUNTEREXAMPLE

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Abstract. The inverse function theorem for analytic functions has a generalization to $C(X)$ which fails in the disk algebra.

Let $A$ be a commutative complex Banach algebra with identity and let $D$ be an open connected subset of $A$. A mapping $\Phi: D \to A$ is analytic in the sense of Lorch (see [3]) if near each point $a \in D$ it has a power series expansion $\Phi(a) = \sum a_n(x-a)^n$. As in classical function theory $a_n = \Phi^{(n)}(a)/n!$. It is known [2] that $\Phi$ has a local analytic inverse mapping $\Phi(a)$ to $a$ precisely when $\Phi'(a)$ is an invertible element of $A$. Classically, when $A$ is the complex numbers, one can say that $\Phi$ has a global analytic inverse if $\Phi$ is simply one to one on $D$. This is not true in general. For example take $A = D = C([0, 1])$ and $\Phi[f](x) = xf(x)$. Since $\Phi'[f](x) = x$, the derivative $\Phi'[f]$ is never invertible.

B. W. Glickfeld has shown [2] that the above example is typical when $A = C(X)$ for some compact Hausdorff space $X$. His theorem is that if $\Phi: D \to C(X)$ is analytic and one to one, then either the inverse of $\Phi$ is analytic or for some $x_0 \in X$ the value $\Phi[f](x_0)$ is constant over all $f \in D$. Glickfeld asks if this result generalizes to other algebras. If we assume $X$ is to be replaced by the maximal ideal space, then in the disk algebra there is a simple counterexample.

Let $B$ be the open unit ball in the disk algebra. Define $\Phi$ on $B$ by $\Phi[f] = (f-s)^2$ where $s(z) = z$. If $\Phi[f] = \Phi[g]$, then for each $z$ in the closed unit disk we have either $f(z) = g(z)$ or $f(z) = 2z - g(z)$. Since $f$ and $g$ are holomorphic in the open unit disk, the permanence of functional relations [1] implies that either $f = g$ or $f = 2s - g$. The second equality cannot hold if $f$ and $g$ both belong to $B$. For if $g \in B$, then by definition of $B$ the norm of $g$ is less...
than one. From \( f = 2s - g \) we get \( \|\| = 2\|s\| - \|g\| = 2 - \|g\| > 1 \), that is, \( f \notin B \). This shows \( \Phi \) is one to one on \( B \).

Clearly \( \Phi \) is analytic and \( \Phi'[f] = 2(f - s) \). Any function \( f \in B \) maps the closed unit disk continuously into itself; by the fixed point theorem there is a point \( z_f \) with \( f(z_f) = z_f \). This gives us \( \Phi'[f](z_f) = 2(f(z_f) - z_f) = 0 \). In other words \( \Phi'[f] \) is never invertible, and \( \Phi \) cannot have an analytic inverse.

Let \( z_0 \) be a point in the closed unit disk, which is the maximal ideal space of the disk algebra. In contrast to Glickfeld’s theorem the set \( \{\Phi[f](z_0) ; f \in B\} \) always contains more than one point; indeed it is an open subset of the plane.

**References**


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