Hurwitz' Theorem

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Abstract. If \([a_0, a_1, a_2, \cdots]\) is the continued fraction for a real number \(x\), and \(p_n/q_n\) the \(n\)th convergent, define \(\theta_n=q_n/p_n-xq_n\). Hurwitz' Theorem asserts that \(\phi_n=\min(\theta_{n-1}, \theta_n, \theta_{n+1})<5^{-1/2}\) whenever \(\phi_n\) is defined. It is the object of this note to provide a simple proof of this fact.

From the well-known relation

\[
\left| x - \frac{p_n}{q_n} \right| + \left| x - \frac{p_{n+1}}{q_{n+1}} \right| = \frac{1}{q_nq_{n+1}}
\]

we obtain

\[
\left( \frac{q_{n+1}}{q_n} \right)^2 \theta_n - \frac{q_{n+1}}{q_n} \theta_{n+1} = 0
\]

and so

\[
\frac{q_{n+1}}{q_n} = \frac{1 \pm (1 - 4\theta_n\theta_{n+1})^{1/2}}{2\theta_n},
\]

with a similar result when \(n\) is replaced by \(n-1\) throughout. Thus

\[
\frac{1 + (1 - 4\theta_n\theta_{n+1})^{1/2}}{2\theta_n} \geq \frac{q_{n+1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n}
\]

\[
\geq 1 + \frac{2\theta_{n-1}}{1 + (1 - 4\theta_n\theta_{n-1})^{1/2}}
\]

\[
= 1 + \frac{1 - (1 - 4\theta_n\theta_{n-1})^{1/2}}{2\theta_n},
\]

or

\[
2\theta_n \leq (1 - 4\theta_n\theta_{n+1})^{1/2} + (1 - 4\theta_n\theta_{n-1})^{1/2}.
\]

Then \(2\phi_n \leq 2(1-4\phi_n^2)^{1/2}\), whence \(\phi_n \leq 5^{-1/2}\). But equality would require that \(\theta_n = \theta_{n-1} = \theta_{n+1} = 5^{-1/2}\), impossible in view of (1).