PERFECT MAPS OF SYMMETRIZABLE SPACES

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Abstract. It is shown that if \( f : X \to Y \) is a perfect map from a symmetrizable space \( X \) onto a space \( Y \), then \( Y \) is metrizable if and only if \( f \) is a coherent map. This fact, together with certain known results, yields the following: Let \( f : X \to Y \) be a perfect map from a Hausdorff symmetrizable space \( X \) onto a space \( Y \); the following are equivalent: (1) \( X \) is metrizable; (2) \( f \) is a regular map; (3) \( f \) is a coherent map; (4) \( Y \) is metrizable.

A topological space \( X \) is said to be symmetrizable if there exists a nonnegative real valued function \( d \) on \( X \times X \), called a symmetric, which satisfies the following three conditions: (1) \( d(a, b) = 0 \) if and only if \( a = b \); (2) \( d(a, b) = d(b, a) \); (3) a subset \( A \) of \( X \) is closed if and only if whenever \( x \in X - A \), then \( d(x, A) > 0 \).

A function \( f : X \to Y \) from a space \( X \) onto a space \( Y \) is said to be coherent if the space \( X \) is symmetrizable via a symmetric \( d \) such that whenever \( \{a_n\} \) and \( \{b_n\} \) are sequences in \( X \) with \( d(a_n, b_n) \to 0 \) and \( f(a_n) \to y \) in \( Y \), then \( f(b_n) \to y \). Coherent maps are closely related to the regular maps of Arhangel’skii [1, p. 133]. Every regular map is a coherent map. The extent of coherent maps may be seen in the following, which is not difficult to prove: Let \( f : X \to Y \) be a function from a symmetrizable space \( X \) onto a metrizable space \( Y \); then, \( f \) is continuous if and only if \( f \) is a coherent map.

A map is perfect if it is closed, continuous and point inverses are compact, i.e., bicontinuous, sets.

Theorem 1. Let \( f : X \to Y \) be a perfect map from a symmetrizable space \( X \) onto a space \( Y \). Then, \( Y \) is metrizable if and only if \( f \) is a coherent map.

Proof. Assume that \( Y \) is metrizable. Let \( \rho \) be a symmetric for \( X \) and \( d \) be a metric for \( Y \). For points \( a \) and \( b \) in \( X \), let \( \sigma(a, b) = \rho(a, b) + d(f(a), f(b)) \). \( \sigma \) is a symmetric compatible with the topology for \( X \). It is easy to verify that \( f \) is a coherent map by virtue of the symmetric \( \sigma \).

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To prove the converse, assume that \( f \) is a coherent map by virtue of a symmetric \( \rho \) for the space \( X \). For \( a, b \in Y \), define \( d(a, b) = \rho(f^{-1}(a), f^{-1}(b)) \). We shall show that the space \( Y \) is symmetrizable via \( d \). Clearly, \( d(a, b) = d(b, a) \) for all \( a, b \in Y \). That \( d(a, b) = 0 \) if and only if \( a = b \) follows easily from the fact that \( f \) is a coherent map and \( Y \) is a \( T_1 \) space. Let \( A \) be a closed subset of \( Y \) and \( y \in Y - A \). If \( d(y, A) = 0 \), then \( \rho(f^{-1}(y), f^{-1}[A]) = 0 \) and there would exist sequences \( \{b_n\} \) in \( f^{-1}(y) \) and \( \{a_n\} \) in \( f^{-1}[A] \) such that \( \rho(a_n, b_n) \to 0 \); since \( f(b_n) \to y \), we would have \( f(a_n) \to y \), contradicting the closedness of \( A \). Consequently, \( d(y, A) > 0 \). Finally, assume that \( B \) is not a closed subset of \( Y \). Since \( f \) is a quotient map, \( f^{-1}[B] \) is not a closed subset of \( X \). Therefore, there exists \( x \in X - f^{-1}[B] \) such that \( \rho(x, f^{-1}[B]) = 0 \). Then \( d(f(x), B) = 0 \), and we have completed the proof that the space \( Y \) is symmetrizable via \( d \).

Let \( A \) be a compact subset of \( Y \) and \( B \) be a closed subset of \( Y \) such that \( d(A, B) = 0 \). Then \( \rho(f^{-1}[A], f^{-1}[B]) = 0 \) and since \( f^{-1}[A] \) is compact, there exists \( x \in f^{-1}[A] \) and sequences \( \{a_n\} \) in \( f^{-1}[A] \) and \( \{b_n\} \) in \( f^{-1}[B] \) such that \( a_n \to x \) and \( \rho(a_n, b_n) \to 0 \). Since \( f \) is a coherent map, we have \( f(b_n) \to f(x) \). Since \( B \) is closed, it follows that \( f(x) \in B \) so that \( A \) and \( B \) are not disjoint. Hence, if \( A \) and \( B \) are disjoint subsets of \( Y \) with \( A \) compact and \( B \) closed, then \( d(A, B) > 0 \). By Theorem 2 of [3], \( d \) is a coherent distance function. The metrizability of \( Y \) now follows by a theorem of Niemytzki and Wilson [5], [8], completing the proof.

If a Hausdorff space \( X \) maps perfectly onto a metrizable space, then \( X \) is metrizable if and only if \( X \) has a \( G_\delta \)-diagonal [2], [6]. Recall also that the perfect image of a metrizable space is metrizable [4], [7]. These facts, together with Theorem 1, yield the following:

**Theorem 2.** Let \( f: X \to Y \) be a perfect map from a symmetrizable Hausdorff space \( X \) onto a space \( Y \). The following are equivalent:

1. \( X \) is metrizable.
2. \( f \) is a regular map.
3. \( f \) is a coherent map.
4. \( Y \) is metrizable.

**Proof.** Assume that \( X \) is metrizable. Then \( Y \) is metrizable. It follows that \( f \) is regular [1, p. 134], so that (1) implies (2). As seen in [3], every regular map is coherent, so that (2) implies (3). (3) implies (4) by Theorem 1. Finally, assume that \( Y \) is metrizable. Then \( X \) is regular; any regular space which maps onto a first countable space by a closed map with first countable point inverses is itself first countable, i.e., \( X \) is a first countable space. This completes the proof since any first countable symmetrizable Hausdorff space has a \( G_\delta \)-diagonal.
The Hausdorff condition on $X$ in Theorem 2 cannot be completely removed since there exist non-Hausdorff compact symmetrizable spaces.

REFERENCES