ODD DIMENSIONAL MANIFOLDS WITH REGULAR CONJUGATE LOCUS

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Abstract. We show that all odd dimensional manifolds, for which the first conjugate locus, with respect to some point, is regular, are homeomorphic to a sphere.

In [2] Warner classifies, almost completely, simply connected, complete Riemannian manifolds $M^n$ for which there exists a point $p \in M^n$ whose first conjugate locus is regular. Almost completely means that he needs one of the following two conditions:

1. The first conjugate point with respect to $p$ in any direction has order $k \geq 2$.
2. All the first conjugate points with respect to $p$ lie at the same distance from $p$.

The purpose of this note is to observe that in the odd dimensional case neither assumption is necessary.

Proposition. Let $M^n$ be a complete simply connected Riemannian manifold, $n$ odd, and suppose there exists $p \in M^n$ such that the first conjugate point with respect to $p$ exists in any direction and has constant order $k$. Then $k = n-1$ and $M^n$ is homeomorphic to $S^n$.

Proof. If we show that $k > 1$ then Warner's condition (1) is satisfied and $n$ odd implies $k = n-1$ and $M^n$ homeomorphic to $S^n$. (See [2].)

Suppose $k = 1$. Let $C(p)$ denote the first conjugate locus with respect to $p$. Then $C(p)$ is a smooth closed submanifold of $M_p$ diffeomorphic to an even $(n-1)$-dimensional sphere and transverse to the lines through the origin in $M_p$ (see [1]). For $x \in C(p)$, $\text{Ker}(d\exp_p)_x$ is orthogonal to the line $\{tx | t \in \mathbb{R}\}$ by Gauss' lemma and therefore has a nontrivial projection on the tangent space $C(p)_x$. In this way we can define a 1-dimensional distribution on $C(p)$, i.e., a 1-dimensional tangent line bundle that is trivial, since $C(p)$ is diffeomorphic to a sphere, and therefore define a nowhere zero vector field. But this is impossible since $n-1$ is even.

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Remark. It is easily seen that a manifold for which there exists a point whose first conjugate locus has order \( n - 1 \) is diffeomorphic to the union of two disks. Conversely, if \( M^n = D^n \cup_\partial D^n \), \( g: \partial D^n \to \partial D^n \) being a diffeomorphism, it is always possible to put a metric on \( M^n \) such that there exists a point whose first conjugate locus has order \( n - 1 \) [2]. Therefore no better result is possible under these hypotheses.

References


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