COMPLETENESS OF EIGENVECTORS
IN BANACH SPACES
HAROLD E. BENZINGER

Abstract. We prove a general theorem on the completeness of
the eigenvectors of linear operators in a Banach space. We then
derive asymptotic estimates for the Green's functions of two-point
boundary value problems which allow us to apply the above theorem
to a wide class of such problems in the spaces $L^p(0, 1)$, $1 \leq p < \infty$.

1. Introduction. Let $B$ denote a Banach space, and let $B^*$ denote its
dual. A sequence $\{\varphi_k\}$ of elements of $B$ is complete in $B$ if the collection of
all finite sums $\sum \alpha_k \varphi_k$, $\alpha_k$ a scalar, is dense in $B$. The sequence $\{\varphi_k\}$ is closed
in $B$ if the only element $\psi$ of $B^*$ for which $\psi(\varphi_k) = 0$, all $k$, is the zero of $B^*$.
It is easily seen that $\{\varphi_k\}$ is closed if and only if $\{\varphi_k\}$ is complete.

For the case that the scalar field is the complex field, we consider the
problem of determining if a sequence $\{\varphi_k\}$ is complete in $B$, where the $\varphi_k$'s
arise as the eigenvectors and generalized eigenvectors of a linear operator
$T: B \to B$. In the case that $B$ is a Hilbert space, there are completeness
results provided that the resolvent operator is a Hilbert-Schmidt operator
or an operator of class $C_p$, and the norm of the resolvent operator obeys
certain growth conditions [1, pp. 1042, 1089, 1115]. These results are
extended to Banach spaces in [9], [10].

If $T: B \to B$ has a compact resolvent $R(\lambda, T)$ for some $\lambda$, then the spectrum
of $T$ is at most countably infinite, consisting entirely of eigenvalues $\lambda_i
which are poles of $R(\lambda, T)$ [8, p. 416]. The invariant subspace corre-
sponding to an eigenvalue $\lambda_i$ is of finite dimension $v_i$. By the operational
calculus [7, pp. 287, 305], the projection $P_i$ of $B$ onto the invariant sub-
space corresponding to $\lambda_i$ has the form

\[
P_i f = \sum_{j=1}^{v_i} \psi_{ij}(f) \varphi_{ij}, \quad f \in B,
\]

where $\varphi_{ij} \in B$, $\psi_{ij} \in B^*$, and

\[
\psi_{ij}(\varphi_{kl}) = \delta_{ik} \delta_{jl}.
\]

In $\S 2$, we shall prove the following result.
Theorem 1.1. Let $T : B \to B$ be a densely defined linear operator with compact resolvent. Then the sequence $\{t_{ik}\}$ is complete in $B$ provided that for each $r > 0$ sufficiently large, the annulus $2r \leq |\lambda| \leq 3r$ contains a circle $C$ centered at the origin, lying entirely in the resolvent set of $T$, such that

$$\|R(\lambda, T)\| \leq K |\lambda|^\mu$$

for $\lambda$ on $C$, where $K$ is a constant, and $\mu$ is an integer.

2. The completeness theorem. Since $T$ is densely defined in $B$, its adjoint $T^* : B^* \to B^*$ is well defined, and is a closed linear operator [6, p. 43]. Since $T$ and $T^*$ have the same resolvent sets we have $R(\lambda, T^*) = R^*(\lambda, T)$ [6, p. 56]. Thus the residue at $\lambda_i$ of $R(\lambda, T^*)$ is the adjoint $P_i^*$ of the residue of $R(\lambda, T)$ at $\lambda_i$, and $P_i^*$ has the form

$$P_i^* g = \sum_{j=1}^{v_i} \varphi_{ij}(g) \psi_{ij}$$

for $g$ in $B^*$. For convenience, we relabel the sequence $\{\varphi_{ij}\}$ as $\{\varphi_k\}$, and similarly for $\{\psi_k\}$. If $\{\varphi_k\}$ is not closed in $B$, there exists a nonzero $g$ in $B^*$ such that $\varphi_k(g) = 0$ for all $k$. For such a $g$, $P^*_i g = 0$ for all $i$. Using the bi-orthogonality (1.2), we see that the converse is true. Consequently, $\{\varphi_k\}$ is closed if and only if the only element $g$ of $B^*$ for which $R(\lambda, T^*) g$ is entire is the zero of $B^*$. See also Definition 3 in [8, p. 443] and the resulting discussion.

Lemma 2.1. Let $T : B \to B$ be a linear operator, and let $f$ be in $B$, $f \neq 0$. Then the equation

$$Tu = \lambda u +$$

has no solution $u(\lambda)$ which is a polynomial in $\lambda$ on an infinite set $S$.

Proof. If we assume that $u(\lambda) = \sum_{k=0}^{m} \lambda^k u_k$, $u_m \neq 0$, is a solution of (2.2) for each $\lambda$ in $S$, then we easily see that each $u_k$ is in the domain of $T$. Substituting this expression into (2.2), we must have $u_m = 0$, a contradiction.

If $\lambda$ is in the resolvent set of $T$, then the unique solution to (2.2) is

$$u(\lambda) = -R(\lambda, T)f.$$ 

Thus any entire solution to (2.2) is an analytic continuation of $-R(\lambda, T)f$ onto the spectrum of $T$.

Proof of Theorem 1.1. Assume $\{\varphi_k\}$ is not closed in $B$. Then there exists an element $g$ in $B^*$, $g \neq 0$, such that $\nu(\lambda) = R(\lambda, T^*) g$ is entire. Let $\lambda_0$ be a fixed complex number, with $|\lambda_0|$ sufficiently large so that the
annulus $2|\lambda_0| \leq |\lambda| \leq 3|\lambda_0|$ contains a circle $C$ on which $||R(\lambda, T^*)|| \leq K|\lambda|^\mu$. Since $v(\lambda)$ is entire,

$$v(\lambda) = (\frac{1}{2}\pi i) \int_C [v(\lambda)/(\lambda - \lambda_0)] d\lambda.$$ 

Since $|\lambda - \lambda_0| \geq |\lambda_0|$, and $|\lambda| \leq 3|\lambda_0|$, we have

$$||v(\lambda)|| \leq (\frac{1}{2}\pi) \int_C [||v(\lambda)||/|\lambda_0|] |d\lambda| \leq 3K |3\lambda_0|^\mu ||g|| = K' |\lambda_0|^\mu.$$ 

Thus $-v(\lambda)$ is a polynomial solution to $T^*v = \lambda v + g$. By Lemma 2.1, this is not possible for $g \neq 0$, so $\{\varphi_k\}$ is closed in $B$.

**Corollary.** If $B$ is reflexive, then under the assumptions of Theorem 1.1, the sequence $\{\psi_k\}$ is complete in $B^*$. 

**Proof.** If $\{\psi_k\}$ is not closed in $B^*$, then there exists $f$ in $B^{**} = B, f \neq 0$, such that $P_i f = 0$ for all $i$. The remainder of the discussion is as in the previous proof.

3. Completeness for ordinary differential operators. Let $l$ denote the $n$th order ordinary linear differential expression defined by

$$(3.1) \quad l(u) = u^{(n)} + a_{n-1}(x)u^{(n-1)} + \cdots + a_0(x)u, \quad 0 \leq x \leq 1,$$

where the $a_j$'s are bounded measurable functions, and in addition $a^{(n-1)}_{n-1}$ exists and is also a bounded measurable function. Let $M, N$ denote two matrices of complex constants with $n$ linearly independent columns between them. Let $u(x)$ denote the column vector $(u(x), u^{(1)}(x), \cdots, u^{(n-1)}(x))$. Let

$$(3.2) \quad Uu = Mu(0) + Nu(1).$$

For $1 \leq p < \infty$, let $\Delta = \Delta_p$ denote the subspace of $L^p(0, 1)$ consisting of all functions $u$ of class $C^{n-1}[0, 1]$ such that $u^{(n-1)}$ is absolutely continuous, $u^{(n)}$ is of class $L^p(0, 1)$, and such that $Uu = 0$. Let $T: L^p \to L^p$ be defined on $\Delta$ by $Tu = l(u)$. Since $\Delta$ contains all functions of class $C^n[0, 1]$, which vanish, along with their first $n-1$ derivatives, at the endpoints, we see that $T$ is densely defined.

If $\lambda$ is in the resolvent set of $T$, then the solution to $Tu = \lambda u + f$ in $L^p(0, 1)$, is

$$(3.3) \quad u(x, \lambda) = \int_0^1 G(x, t, \lambda)f(t) dt = -R(\lambda, T)f,$$

where $G$ is the Green's function of $T$.

Since $a^{(n-1)}_{n-1}$ is in $L^\infty [0, 1]$, we can perform a substitution $u(x) = q(x)v(x)$,
where
\[ q(x) = \exp\left[ -(1/n) \int_0^x a_{n-1}(t) \, dt \right], \]
and obtain a new differential expression and boundary conditions for \( v \).

The significant feature of the transformed problem is that the coefficient of \( v^{(n-1)} \) is zero. This simplifies the discussion of the asymptotic nature of solutions to \( l(u) = \lambda u \).

**Definition 3.1.** The differential operator \( T \) is *Stone regular* if the transformed problem satisfies Definition 3.1 in [2, p. 487].

If \( G'(x, t, \lambda) \) denotes the Green's function of the transformed problem, then as observed in [3, equation 2.5],
\[ G(x, t, \lambda) = q(x)G'(x, t, \lambda)q^{-1}(t), \]
where in [3] we used the substitution \( \lambda = -\rho^n \). Thus we shall dispense with the distinction between the original problem and its transformed version, and assume that \( a_{n-1} = 0 \).

The location of the eigenvalues of \( T \) is discussed in [2, p. 489]. It is convenient for this purpose to refer to the \( \rho \)-plane. We will use the notation of [2], in particular the sectors \( S_i \) are defined on p. 483, and the constants \( \sigma \) and \( \tau \) are defined on p. 485. Let \( \delta > 0 \) be given. It is clear from the discussion in [2] that if each \( \rho \in S_i \) such that \(-\rho^n \) is an eigenvalue of \( T \) is centered at a disc of radius \( \delta \), then for \( r > 0 \) sufficiently large, each region in \( S_i \) of the form \((2r)^{1/n} \leq |\rho| \leq (3r)^{1/n}\) contains many circular arcs centered at the origin of the \( \rho \)-plane, and not intersecting any of the discs. The image in the \( \lambda \)-plane of such an arc is a circle \( C \), centered at the origin of the \( \lambda \)-plane, contained entirely in the resolvent set of \( T \), and satisfying \( 2r \leq |\lambda| \leq 3r \).

**Theorem 3.1.** If the differential operator \( T \) is Stone regular, there exists an integer \( m \geq 0 \) such that
\[ n\rho^{n-1}G(x, t, \rho) = \rho^m O(1) \]
as \( |\rho| \to \infty \) in \( S'_i \) where the \( O(1) \) term is uniform in \( t \) and \( x \) for \( 0 \leq t, x \leq 1 \).

**Proof.** This is a direct consequence of equations (2.9) and (4.7) in [2, pp. 484, 492].

**Corollary.** If \( \lambda = -\rho^n \) is in the resolvent set of \( T \), and if \( |\rho - \rho_0| \geq \delta \) for each eigenvalue \( \lambda_0 = -\rho_0 \), then for \( |\lambda| \) sufficiently large,
\[ |G(x, t, \lambda)| \leq K |\lambda|^{(m+1-n)/n}, \quad 0 \leq t, x \leq 1, \]
where \( K \) is a constant.

**Proof.** This is a direct consequence of equations (3.4) and (3.5).
We note at this point that there is no theoretical limit to the size of \( m \). See [2, Theorem 5.3]. Let \( \mu \) denote the first integer no smaller than \((m+1-n)/n\).

**Theorem 3.2.** If \( T \) is Stone regular, then for each \( \lambda = -\rho^n \) in the resolvent set of \( T \) such that \(|\rho - \rho_0| \geq \delta \) for each eigenvalue \( \lambda_0 = -\rho_0^n \), as an operator from \( L^p \) to \( L^p \),

\[(3.7) \quad \|R(\lambda, T)\| \leq K|\lambda|^\mu.\]

**Proof.** By (3.6), we see that \( G \), as a function to \( t \), is of class \( L^\infty(0, 1) \), for fixed \( x \) and \( \lambda \). Thus \( G \) is in \( L^q(0, 1) \) for each \( q, 1 \leq q \leq \infty \). If \( f \) is in \( L^p(0, 1), 1 \leq p < \infty \), and if \( p+q = pq \), then by Hölder's inequality,

\[
|u(x, \lambda)| \leq \left[ \int_0^1 |G(x, t, \lambda)|^q \, dt \right]^{1/q} \|f\|_p \leq K|\lambda|^\mu \|f\|_p.
\]

Thus \( \|u(\cdot, \lambda)\|_p \leq K|\lambda|^\mu \|f\|_p \).

**Remark.** In particular, (3.7) holds on each circle \( C \) which is the image of a circular arc lying entirely in \( S' \).

**Theorem 3.3.** If \( T \) is Stone regular, the eigenfunctions and generalized eigenfunctions of \( T \) form a sequence which is complete in \( L^p(0, 1) \) for \( 1 \leq p < \infty \).

**Corollary.** The eigenfunctions and generalized eigenfunctions of \( T^* \) are complete in \( L^p(0, 1) \) for \( 1 < p < \infty \).

**Remark.** The adjoint in \( L^p \) of a two-point boundary value problem in \( L^p \) is known to be another two-point boundary value problem \((1 < p < \infty)\), provided that the coefficients \( a_j \) are sufficiently differentiable [5], so in such a case the corollary provides no new information. If the \( a_j \)'s are not sufficiently differentiable, the \( L^p \) adjoint of \( T \) is a quasi-differential operator [4, p. 888]. Thus in these cases the corollary provides new information.

**References**


**Department of Mathematics, University of Illinois, Urbana, Illinois 61801**