A CHARACTERIZATION OF THE JACOBSON RADICAL IN TERNARY ALGEBRAS

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Abstract. The Jacobson radical Rad T for a ternary algebra T is characterized as one of the following: (i) the set of properly quasi-invertible elements in T; (ii) the set of x ∈ T such that the principal right ideal ⟨xTT⟩ or left ideal ⟨TTx⟩ is quasi-regular in T; (iii) the unique maximal quasi-regular ideal in T; (iv) the set of x ∈ T such that Rad T⟨x⟩ = T⟨x⟩. We also obtain ternary algebra-analogs of characterization of the radicals of certain subalgebras in an associative algebra.

1. Introduction. Let Φ be a commutative ring with identity. A ternary algebra (τ-algebra) over Φ is defined as a unital Φ-module T with a trilinear composition (x, y, z) → ⟨xyz⟩ satisfying

\[ \langle(xy)uv \rangle = \langle x(yzu)v \rangle = \langle xy(zuv) \rangle. \]

In case Φ = Z, τ-algebras are τ-rings defined by Lister [1], who extensively studies ring-theoretic structures for τ-rings, in particular, a module theory leading up to the Jacobson radical. Lister defines the Jacobson radical, R(T), for a τ-ring T to be the intersection of kernels of its irreducible modules and shows that R(T) is the intersection of all maximal modular ideals in T [1, Theorem 9]. Let T now be a τ-algebra. For a ∈ T, we denote by T(a) the algebra formed by setting

\[ x \cdot y = \langle xay \rangle \]

on the module T. Then, by (1), T(a) is an associative algebra. An element x ∈ T is called properly quasi-invertible (p.q.i.) in T if it is quasi-invertible (q.i.) in T(a) for all a ∈ T.

Definition. For a τ-algebra T, the Jacobson radical, Rad T, of T is the set of all p.q.i. elements in T.

In this paper we establish τ-algebra-analogs of characterization of the radical in associative or Jordan algebras. Specifically, we will show that Rad T is an ideal of T and coincides with the following: (i) the set of...
$x \in T$ such that \( \langle xTT \rangle \) (or \( \langle TTx \rangle \)) is a quasi-regular right (or left) ideal in $T$; (ii) $\bigcap_{x \in T} \text{Rad } T^{(a)}$; (iii) the set of $x \in T$ such that $\text{Rad } T^{(a)} = T^{(a)}$. As applications of this we obtain $\tau$-algebra-analogs of such results as Theorems 4, 6, and 7 given by McCrimmon in [3]. Also, the present radical $\text{Rad } T$ coincides with $R(T)$ if $T$ is a $\tau$-ring of Lister.

$\tau$-algebras have been called associative triple systems of 1st kind by Loos [2], who has obtained analogous results for associative triple systems of 2nd kind, defined by the rule

$$
\langle \langle xyz \rangle uv \rangle = \langle x(uzy)v \rangle = \langle xy(zuv) \rangle.
$$

The basic example of the 1st kind (or $\tau$-algebra) is a subspace of an associative algebra that is closed relative to $\langle xyz \rangle = xyz$, while the basic example of the 2nd kind is a subspace of an associative algebra with involution, which is closed relative to $\langle xyz \rangle = x\bar{y}z$. Setting $P(x)y = \langle xyx \rangle$, the 1st and 2nd kinds together with the quadratic mapping $P$ become Jordan triple systems (JTS) introduced by K. Meyberg [4]. Some of our present results have been proved for JTS in [4] where its proofs use complicated identities.

2. Preliminaries. Throughout we assume $T$ denotes a $\tau$-algebra over $\Phi$. Any $\tau$-algebra $T$ can be imbedded into an associative algebra $A = T + T^2$ regarded as a $\tau$-algebra [1, p. 39]. Then $A$ is called an enveloping algebra of $T$ and if $A = T \oplus T^2$, the imbedding is called direct. A $\tau$-algebra always possesses a direct enveloping algebra [1, p. 38]. If $A = T + T^2$ and $a \in A$, we also denote by $A^{(a)}$ the $a$-homotope of $A$. Thus if $a \in T$, $T^{(a)}$ is a subalgebra of $A^{(a)}$. Recall $x \in A$ is q.i. in $A$ if $x + y = xy = yx$ for some $y \in A$.

For $x, y \in T$ we introduce the useful operator

$$
B(x, y) \equiv \text{Id} - L(x, y): z \mapsto \langle xyz \rangle
$$

for $z \in T$. Then we directly see from (1) the following:

(2) $B(\langle xyz \rangle, u) = B(x, \langle yzu \rangle),$

(3) $B(x, y)B(x, -y) = B(x, \langle yxy \rangle).$

If $x, y \in T$ and $y$ is a quasi-inverse of $x$ in $T^{(u)}$, then $x + y = \langle xuy \rangle$ and for all $z \in T$,

$$
B(x, u)B(-y, z)v = v + \langle yzv \rangle - \langle xuv \rangle - \langle x\langle yzv \rangle \rangle
$$

$$
= v + \langle yzv \rangle - \langle xuv \rangle - \langle (x + y)zv \rangle \quad \text{(by (1))}
$$

$$
= B(x, u + z)v.
$$

Hence we have the Addition Formula: if $x, y \in T$ are quasi-inverses in $T^{(u)}$, $B(x, u)B(-y, z) = B(x, u + z)$ for all $z \in T$. 

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Lemma 1. If \( A = T \oplus T^2 \), then, for \( a, x \in T \), the following are equivalent:

(i) \( x \) is q.i. in \( T^{(a)} \);
(ii) \( x \) is q.i. in \( T^{(x)} \);
(iii) \( a \) is q.i. in \( T^{(a)} \);
(iv) \( a \) is q.i. in \( T^{(x)} \);
(v) \( x a \) is q.i. in \( A \);
(vi) \( a x \) is q.i. in \( A \);
(vii) \( B(x, a) \) is bijective on \( T \);
(viii) \( B(a, x) \) is bijective on \( T \).

Proof. That (i) \( \Rightarrow \) (ii) is obvious, and that (ii) \( \Rightarrow \) (iii) is a result of McCrimmon [3, Proposition 1]. (iii) \( \Rightarrow \) (iv): since \( x a \) is q.i. in \( A \), the left multiplication \( L(1 - xa) \) in \( A \) is invertible on \( A \) (if \( A \) does not contain 1, adjoin an identity to \( A \)). Then \( T \) and \( T^2 \) are invariant under \( L(1 - xa) \), and \( B(x, a) \) is the restriction of \( L(1 - xa) \) to \( T \). Noting that an element in \( T^2 \) is q.i. in \( T^2 \) if and only if it is q.i. in \( A \) since the imbedding is direct, we have that \( B(x, a) \) is still invertible on \( T \). (iv) \( \Rightarrow \) (i): by surjectivity \( B(x, a)y = -x \) for some \( y \in T \), i.e., \( x + y = \langle xay \rangle \). But

\[
B(x, a)\langle xay \rangle = \langle yax \rangle - \langle xa\langle yax \rangle \rangle = \langle yax \rangle - \langle (xay)ax \rangle
\]

\[
= \langle yax \rangle - \langle xax \rangle - \langle yax \rangle = -\langle xax \rangle
\]

\[
x + y - \langle xa(x + y) \rangle = B(x, a)(x + y)
\]

and so \( x + y = \langle yax \rangle \) too since \( B(x, a) \) is injective. By symmetry we prove that (i') \( \Leftrightarrow \) (ii') \( \Leftrightarrow \) (iii') \( \Leftrightarrow \) (iv'). But that (iii) \( \Leftrightarrow \) (iii') is well known for associative algebras, and the proof is complete.

By symmetry in Lemma 1 in \( x \) and \( a \), we have a Symmetry Principle for \( T \):

\( x \) is q.i. in \( T^{(y)} \) if and only if \( y \) is q.i. in \( T^{(x)} \).

From this we get a Shifting Principle for \( T \):

\( x \) is q.i. in \( T^{(xyzu)} \) if and only if \( u \) is q.i. in \( T^{(xyz)} \),

since, by Lemma 1, \( x \) is q.i. in \( T^{(xyzu)} \) if and only if \( B(x, \langle yzu \rangle) = B(\langle xyz \rangle, u) \) (by (2)) is bijective if and only if \( u \) is q.i. in \( T^{(xyzu)} \). Hence we obtain the Shifting Theorem for \( T \):

Theorem 1. The following conditions for \( x, y, z, u \) in a \( r \)-algebra \( T \) are equivalent:

(i) \( x \) is q.i. in \( T^{(yuz)} \);
(ii) \( u \) is q.i. in \( T^{(xyzu)} \);
(iii) \( y \) is q.i. in \( T^{(xzu)} \);
(iv) \( z \) is q.i. in \( T^{(xyzu)} \).
In view of the equivalence of (i), (ii), and (iii') in Lemma 1 we can state

**Lemma 2.** If \( A = T \oplus T^2 \), then \( x \in T \) is p.q.i. in \( T \) if and only if \( xa \) and \( ax \) are q.i. in \( T^2 \) (or in \( A \)) for all \( a \in T \).

3. **Characterization of the radical.** A subspace \( V \) of \( T \) is called a **right** ideal of \( T \) in case \( \langle VTT \rangle \subseteq V \), a **left** ideal in case \( \langle TT^2 \rangle \subseteq V \), a **medial** ideal in case \( \langle TTV \rangle \subseteq V \), and \( V \) is called an **ideal** in \( T \) if it is right, left, and medial. Thus a right (left) ideal \( V \) in \( T \) is a right (left) ideal in all \( T^{(a)} \). A right ideal \( V \) of \( T \) is called (right) **quasi-regular** (q.r.) \([1, \text{p. 45}]\) if \( B(v, x)T = T \) for all \( v \in V \) and all \( x \in T \), and a q.r. left ideal is similarly defined. Noting that \( V \) is (right) q.r. in all \( T^{(a)} \), this condition is equivalent to that every \( v \in V \) is p.q.i. in \( T \).

**Theorem 2.** For a \( \tau \)-algebra \( T \), the radical \( \text{Rad} \ T \) is an ideal of \( T \).

**Proof.** Let \( x \in T \) be p.q.i. in \( T \). Then \( ax \) is p.q.i. for \( a \in \Phi \) since all \( B(ax, y) = B(x, ay) \) are bijective. Now, \( \langle xyz \rangle \) are p.q.i. for all \( x, y, z \in T \) since \( B((xyz), u) = B(x, (yzu)) \), and all \( \langle yxz \rangle \) are p.q.i. since all \( B(y, (xzu)) \) are bijective (Symmetry) and so are all \( B(u, (xyz)) \) (Shifting). From this, using Symmetry and Shifting Principles, we also see that all \( \langle yzx \rangle \) are p.q.i. in \( T \). Finally, let \( z, u \) be p.q.i. in \( T \) and let \( x \in T \). By Symmetry \( x \) is q.i. in \( T^{(u)} \), and if \( y \) is the quasi-inverse of \( x \) in \( T^{(u)} \) then by the Addition Formula \( B(x, u)B(-y, z) = B(x, u+z) \). Since \( B(x, u) \) and \( B(-y, z) \) are bijective so is \( B(x, u+z) \) for all \( x \in T \). Thus the set of all p.q.i. elements in \( T \) forms an ideal of \( T \).

**Lemma 3.** If \( A = T \oplus T^2 \), the following are equivalent:

(i) \( x \in T \) is p.q.i. in \( T \);
(ii) \( xT \) is a q.r. right ideal in \( T^2 \);
(ii') \( Tx \) is a q.r. left ideal in \( T^2 \);
(iii) the principal right ideal \( \langle xTT \rangle \) is q.r. in \( T \);
(iii') the principal left ideal \( \langle TTx \rangle \) is q.r. in \( T \).

**Proof.** (i) \( \Leftrightarrow \) (ii). This follows from Lemma 2. (i) \( \Leftrightarrow \) (iii). If \( x \) is p.q.i. in \( T \), all \( B(x, (yzu)) = B((xyz), u) \) are bijective and so by Theorem 2 all elements in \( \langle xTT \rangle \) are p.q.i., i.e., \( \langle xTT \rangle \) is q.r. in \( T \). Conversely, if \( \langle xTT \rangle \) is q.r. in \( T \), all \( B((xyz), u) = B(x, (yzu)) \) are bijective. Then by (3) all \( B(x, t)B(x, -t) \) are bijective so all \( B(x, t) \) are too. Since (i) is left-right symmetric, we get (i) \( \Leftrightarrow \) (ii') \( \Leftrightarrow \) (iii'), and the proof is complete.

In view of Theorem 2 and Lemma 3(i), (iii), (iii'), we have the following analogous result of associative algebras (also see [1]):

**Corollary 1.** \( \text{Rad} \ T \) is a q.r. ideal in \( T \) and contains all q.r. right ideals and q.r. left ideals in \( T \). Hence \( \text{Rad} \ T \) is the unique maximal q.r. ideal in \( T \).
Remark. In case $T$ is a $\tau$-ring, this corollary has been proved for the radical $R(T)$ at characteristic $\neq 2$ by Lister [1, Theorem 10]. Hence $\text{Rad } T$ coincides with the radical $R(T)$ defined by Lister.

Corollary 2. If $A = T \oplus T^2$, then
\[
\text{Rad } T = \{ x \in T \mid xT \subseteq \text{Rad } T^2 \}.
\]

Proof. This follows from Lemma 3 and the well-known fact for associative algebras.

Theorem 3. For a $\tau$-algebra $T$, we have
\[
\text{Rad } T = \{ x \in T \mid \langle xTT \rangle \text{ or } \langle TTx \rangle \text{ is q.r. in } T \}
= \{ x \in T \mid \text{Rad } T^{(x)} = T^{(x)} \}
= \bigcap_{a \in T} \text{Rad } T^{(a)}.
\]

Proof. $\text{Rad } T$ equals the first set by Lemma 3(i), (iii), (iii'). If $x \in \text{Rad } T$, $x$ is q.i. in all $T^{(axx)}$ and by Symmetry all $\langle zxx \rangle$ are q.i. in $T^{(x)}$; so are all $z$ in $T^{(x)}$ since $\langle zxx \rangle = z^2$ in $T^{(x)}$. Hence $\text{Rad } T^{(x)} = T^{(x)}$ conversely, if $\text{Rad } T^{(x)} = T^{(x)}$, all $y$ are q.i. in $T^{(x)}$ and by Symmetry $x$ is p.q.i. in $T$, i.e., $x \in \text{Rad } T$. Thus $\text{Rad } T$ equals the second set. If $x \in \bigcap \text{Rad } T^{(a)}$, $x$ is q.i. in all $T^{(a)}$ and is p.q.i. in $T$, while if $x$ is p.q.i. in $T$, so is $x$ in all $T^{(a)}$ since the $y$-homotope of $T^{(a)}$ is $T^{(axa)}$. Hence $\text{Rad } T = \bigcap \text{Rad } T^{(a)}$. This completes the proof.

4. Applications. In this section we use the previous results to characterize the radicals of certain $\tau$-subalgebras of $T$. A left (or right, or left-right) ideal $V$ of $T$ is also a left (or right, or two-sided) ideal in all $T^{(a)}$, and $V^{(a)}$ for every $a \in T$ is a left, right, or two-sided ideal of $T^{(a)}$, according as $V$ is left, right, or left-right. Hence, in any case, $V$ is a strict inner ideal in all $T^{(a)}$ and $V^{(a)}$ is strictly inner in $T^{(a)}$. We now recall that if $K$ is a strict inner ideal of an alternative algebra $B$ and $x \in K$ is q.i. in $B$, then $x$ is q.i. in $K$ [3, p. 571]. If $U$ is an ideal in a $\tau$-ring $T$, Lister [1, p. 46] proves that $R(U) = U \cap R(T)$. We extend this to left-medial (or medial-right) ideals in a $\tau$-algebra $T$ and give another proof of this as well.

Theorem 4. If $V$ is a left-medial (or medial-right) ideal in $T$, then $\text{Rad } V = V \cap \text{Rad } T$.

Proof. We proceed as in [3, Theorem 3]. If $x \in V \cap \text{Rad } T$, $x$ is q.i. in all $T^{(a)}$ and so q.i. in all $V^{(a)}$ since $V^{(a)}$ is a left ideal in $T^{(a)}$. Hence $x$ is p.q.i. in $V$ and so $x \in \text{Rad } V$. Conversely, if $x \in \text{Rad } V$, $x$ is q.i. in $V^{(axa)}$ (so in $T^{(axa)}$) for all $a \in T$ since $V$ is medial. Hence, by Symmetry, each $\langle axa \rangle$, which is the square of $a$ in $T^{(x)}$, is q.i. in $T^{(x)}$. Thus all $a$ are q.i. in $T^{(x)}$ and by Symmetry $x \in \text{Rad } T$. 
For left or right ideals of $T$, we obtain a result similar to [3, Theorem 4]:

**Theorem 5.** If $V$ is a left ideal in $T$, then

\[
\text{Rad } V = \{ z \in V \mid \langle VVz \rangle \subseteq \text{Rad } T \} = \{ z \in V \mid \langle TVz \rangle \subseteq \text{Rad } T \}.
\]

**Proof.** We only prove the second equality and the other case is entirely similar. Let $z \in V$ be such that $\langle TVz \rangle \subseteq \text{Rad } T$. Then every $x \in \langle TVz \rangle$ is q.i. in all $T^{(a)}$ and since $V^{(a)}$ is a left ideal of $T^{(a)}$, $x$ is q.i. in all $V^{(a)}$, so in particular $\langle VVz \rangle$ is a q.r. left ideal of $V$. Hence by Lemma 3, $z \in \text{Rad } V$. Conversely, if $z \in \text{Rad } V$, let $a$ be any element of $V$. Then $z$ is q.i. in $V^{(a+yaz)}$ (so in $T^{(a+yaz)}$) for all $x, y \in T$, and so, shifting, all $x$ are q.i. in $T^{(a+yaz)}$. Hence by Symmetry $\langle yaz \rangle$ is q.i. in all $T^{(a)}$ and $\langle yaz \rangle \in \text{Rad } T$. Thus $\langle TVz \rangle \subseteq \text{Rad } T$.

As usual, a subspace $V$ of $T$ is called an *inner ideal* of $T$ if $\langle aTa \rangle \subseteq V$ for all $a \in V$. Thus all left, right ideals and the subspaces $\langle xTx \rangle$ are inner ideals in $T$. Furthermore, if $V$ is an inner ideal in $T$ then $V^{(a)}$ is a strict inner ideal in $T^{(a)}$ for all $a \in T$ since $\langle vav \rangle = v^2$ in $V^{(a)}$ for $v \in V$. The following characterization of the radical is applied for all these examples.

**Theorem 6.** If $V$ is an inner ideal in $T$, then

\[
\text{Rad } V = \{ z \in V \mid \langleaza \rangle \in \text{Rad } T \text{ for all } a \in V \}.
\]

**Proof.** If $z \in V$ and $\langleaza \rangle \in \text{Rad } T$ for all $a \in V$, $\langleaza \rangle \in V$ is q.i. in all $T^{(z)}$ and so is q.i. in all $V^{(z)}$ since $V^{(z)}$ is strictly inner in $T^{(z)}$. In particular, all $\langleaza \rangle$ are q.i. in $V^{(z)}$, but $\langleaza \rangle = az^2$ in $V^{(z)}$ and so all $a$ are q.i. in $V^{(z)}$ (again recall $V^{(z)}$ is strictly inner in $T^{(z)}$). Thus by Symmetry $z \in \text{Rad } V$. The converse is the same as in [3, Theorem 7].

An element $e \in T$ is called an *idempotent* if $\langle eee \rangle = e^3 = e$. If $z \in \text{Rad } \langle eTe \rangle$ then $z = \langle exe \rangle$ for some $x \in T$ and so $z = \langle e(eee)e \rangle \in \text{Rad } T$ by Theorem 6. Hence we obtain

**Corollary.** If $e$ is an idempotent in $T$, then

\[
\text{Rad } \langle eTe \rangle = \langle eTe \rangle \cap \text{Rad } T.
\]

5. **Relations with Jordan triple systems.** For a $\tau$-algebra $T$, let $T_J$ denote the JTS formed from $T$ by setting $P(a)x = \langle axa \rangle$. From $T^{(a)}$, we also form a quadratic Jordan algebra $T^{(a)+}$ by setting $P_a(x) = P(x)P(a)$ and $x^{2(a)} = P(a)x$, and it is well known that $\text{Rad } T^{(a)} = \text{Rad } T^{(a)+}$. Thus, in view of Theorem 3 and [4], we have

**Theorem 7.** For a $\tau$-algebra $T$, $\text{Rad } T = \text{Rad } T_J$.  

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As usual, $a \in T$ is called von Neumann regular if $a = \langle axa \rangle$ for some $x \in T$, and $b \in T$ is called trivial if $\langle bTb \rangle = 0$. While one-sided ideals or ideals in $T$ are different from those in $T_J$, $T$ and $T_J$ have the same inner ideals, idempotents, trivial elements, and von Neumann regularity. Hence, from Theorem 7 and the known result [4] for a JTS, we can state

**Theorem 8.** If $T$ is a $\tau$-algebra satisfying the d.c.c. on inner ideals, then the following are equivalent:

(i) $T$ is semisimple;
(ii) $T$ is von Neumann regular;
(iii) $T$ contains no nonzero trivial elements.

A proof of this for a JTS uses complicated identities [4], but some parts of the proof for $T$ are considerably shorter. For example, to show that if $a \in T$ is trivial then $a \in \text{Rad } T$, we simply observe that every element of $\langle aTT \rangle$ is trivial and so $\langle aTT \rangle$ is q.r. in $T$. Next, to see that if $T$ is von Neumann regular then it is semisimple, let $z \in \text{Rad } T$. Then $z = \langle zaz \rangle$ for some $a$, so $B(z, a)z = 0$; but since $B(z, a)$ is invertible, $z = 0$.

Finally, let $B$ be an associative algebra. Then $B$ becomes a $\tau$-algebra on iteration of its product, which we denote by $B_\tau$. Let $B^+$ and $B_J$, respectively, denote the quadratic Jordan algebra and the JTS formed from $B$ in the usual manner. Then from Theorems 3 and 7 we obtain

**Theorem 9.** If $B$ is an associative algebra, then

$$\text{Rad } B = \text{Rad } B^+ = \text{Rad } B_\tau = \text{Rad } B_J.$$

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**References**


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