THE FUNDAMENTAL IDEAL AND $\pi_2$ OF HIGHER DIMENSIONAL KNOTS

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Abstract. Let $(S^4, k(S^3))$ be a knot formed by spinning a polyhedral arc $\alpha$ about the standard 2-sphere $S^2$ in the 3-sphere $S^3$. Then the second homotopy group of $S^4 - k(S^3)$ as a $\mathbb{Z}\pi_1$-module is isomorphic to each of the following:

1. The fundamental ideal modulo the left ideal generated by $a - 1$, where $a$ is the image in $\pi_1(S^4 - k(S^3))$ of a generator of $\pi_1(S^2 - \alpha)$.

2. The first homology group of the kernel of $\pi_1(S^4 - k(S^3)) \to \pi_1(S^4 - k(S^3))$.

0. Introduction. A presentation of the second homotopy group of an arbitrary spun knot [3] was calculated as a $\mathbb{Z}\pi_1$-module in [4], [5], [6]. Professor John Milnor has conjectured that this is a presentation of the fundamental ideal modulo a suitably chosen element. In this paper, we show that this is actually the case. In particular,

THEOREM 2. Let $k(S^2)$ be a 2-sphere formed by spinning an arc $\alpha$ about the standard 2-sphere $S^2$ in the 3-sphere $S^3$. Let $a$ denote the image in $\pi_1(S^4 - k(S^3))$ of a generator of $\pi_1(S^2 - \alpha)$ and let $\mathfrak{I}$ be the fundamental ideal in $\mathbb{Z}\pi_1$, i.e., the two sided ideal generated by all elements of the form $g - 1$ ($g \in \pi_1$). Then the second homotopy group $\pi_2(S^4 - k(S^3))$ as a $\mathbb{Z}\pi_1$-module is $\mathfrak{I}/(a - 1)$, where $(a - 1)$ denotes the left ideal generated by $a - 1$. The action of $\mathbb{Z}\pi_1$ on $\pi_2$ is that induced by left multiplication on $\mathfrak{I}$.

Moreover, we have yet another characterization of $\pi_2$, namely

THEOREM 1. Let $(S^4, k(S^3))$ be defined as in Theorem 2 above. Then $\pi_2(S^4 - k(S^3))$ as a $\mathbb{Z}\pi_1$-module is the first homology group of the kernel of $\pi_1(S^4 - k(S^3)) \to \pi_1(S^4 - k(S^3))$.

I. Definition of a spun knot. Let $S^2$ be a standard 2-sphere in the 3-sphere $S^3$ and let $\alpha$ be a polyhedral arc with endpoints lying on $S^2$ and with interior lying entirely within one of the two components of $S^3 - S^2$. 
Generate a knotted 2-sphere \( k(S^2) \) in \( S^4 \) by spinning \( \alpha \) about \( S^2 \) while holding \( S^2 \) fixed. (For details see [3], [7].)

II. **Proofs of theorems.** Let \( X = S^4 - k(S^2) \) and let \( X_0 = S^3 - k(S^2) \) be the 3-dimensional cross-section of the knot. Let \( X_+ \) and \( X_- \) denote the closures of the two components of \( X - X_0 \). Similarly, let \( X_{0+} = S^2 - k(S^2) \) and \( X_{0-} \) be the closures of the two components of \( X_0 - X_{00} \). Let \( \tilde{X} \) be the universal covering of \( X \), and \( \tilde{X}_+ \), \( \tilde{X}_0 \), \( \tilde{X}_{0+} \), and \( \tilde{X}_{0-} \) be the respective lifts of \( X_+ \), \( X_0 \), \( X_{0+} \), \( X_{0-} \) to \( \tilde{X} \).

The lifts \( \tilde{X}_+ \), \( \tilde{X}_0 \), \( \tilde{X}_{0\pm} \) are all connected. This can be seen by inspecting the homotopy sequence for the fibration

\[
\pi_1(X) \to \tilde{X}_i \to X_i
\]

and noting that \( \pi_1(X_i) \to \pi_1(X) \) is onto for \( i = +, - \), \( 0, 0+ \), \( 0- \). Moreover, since \( \pi_1(X_i) \to \pi_1(X) \) for \( i = +, - \), \( 0, 0+ \), \( 0- \) is an isomorphism onto [3], \( \tilde{X}_+ \), \( \tilde{X}_{0\pm} \) are all simply connected.

Since \( \tilde{X}_\pm \) collapses to \( \tilde{X}_{0+} \) via a deformation arising from the spinning, Hurewicz's theorem coupled with the asphericity of knots [2] gives \( H_n(\tilde{X}_\pm) = 0 \) for \( n \geq 1 \). Hence, from the Mayer-Vietoris sequence for the triad \( (\tilde{X}; \tilde{X}_+; \tilde{X}_-) \), we have

\[
H_2(\tilde{X}) \cong H_1(\tilde{X}_{0+}).
\]

But by the asphericity of knots [2], \( \tilde{X}_0 \) is the Eilenberg-Mac Lane space \( K(\pi_1(\tilde{X}_0), 1) \). Inspecting the homology sequence for the fibration \( \pi_1(X) \to \tilde{X}_0 \to X_0 \), we have

**Theorem 1.** The second homotopy group \( \pi_2(S^4 - k(S^2)) \) as a \( \mathbb{Z}_{\pi_1} \)-module is the first homology group of the kernel of \( \pi_1(S^3 - k(S^2)) \to \pi_1(S^4 - k(S^2)) \).

Theorem 2 now follows from a close inspection of the exact sequence

\[
0 \to H_1(\tilde{X}_0) \to H_0(\tilde{X}_{00}) \to H_0(\tilde{X}_{0+}) \oplus H_0(\tilde{X}_{0-})
\]

arising from the Mayer-Vietoris sequence for \( (\tilde{X}_0; \tilde{X}_{0+}, \tilde{X}_{0-}) \).

Note that there is a deformation retraction of \( X_{0+} \) onto a 2-dimensional CW-complex \( K_{0+} \) which induces on \( X_{00} \) a deformation retraction onto a 1-dimensional CW-complex \( K_{00} \). The CW-complex \( K_{00} \) consists of a single 1-simplex \( \xi_0 \) and a single vertex \( p \). The element \( a \) of \( \pi_1(X) \) carried by \( \xi_0 \) is the image of a generator of \( \pi_1(\tilde{X}_0) \). (For details see the lemma of [6].)

Thus, \( \tilde{X}_{00} \) collapses to

\[
\tilde{K}_{00} = \bigcup_{g \in \pi_1} g \xi_0.
\]

By direct computation, \( Z_0(\tilde{K}_{00}) = \mathbb{Z}_{\pi_1} p \) and \( B_0(\tilde{K}_{00}) = (a-1)p \), where
(a-1) is the left ideal generated by a-1. Hence, \( H_0(\hat{X}_0) \cong \mathbb{Z}\pi_1/(a-1) \) and
\[
0 \to H_1(\hat{X}_0) \to \mathbb{Z}\pi_1/(a-1) \xrightarrow{\sigma_*} J \oplus J.
\]

Again by direct computation, \( \sigma_* = e \oplus (-e) \), where \( e: \mathbb{Z}\pi_1/(a-1) \to J \) is the projection of the trivializer \([1]\) (also called the augmentation \([8]\)). Thus, \( \ker(\sigma_*) = \ker(e) \cap \ker(-e) = \mathbb{Z}/(a-1) \) and
\[
\mathbb{Z}/(a-1) \cong H_1(\hat{X}_0) \cong H_2(\hat{X}) \cong \pi_2(X).
\]

References

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