ANOSOV DIFFEOMORPHISMS ON NILMANIFOLDS

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ABSTRACT. The purpose of this paper is to give necessary conditions on the map induced by an Anosov diffeomorphism of a nilmanifold on its fundamental group.

We shall generalize a result of Franks [3] which may be rephrased:

THEOREM (FRANKS). Let $T^n$ be the n-dimensional torus and $f: T^n \to T^n$ an Anosov diffeomorphism. Then $f_* : \pi_1(T^n) \to \pi_1(T^n)$ has no roots of unity as eigenvalues.

$M$ will always denote a nilmanifold, that is a compact homogeneous space $N/D$ where $N$ is a connected simply connected nilpotent Lie group and $D$ is a uniform discrete subgroup of $N$. Then $\pi_1(M) = D$. Parry describes in [8] how to use the lower central series of $N$ to express $N/D$ as a sequence of torus extensions. We do the same thing with the upper central series of $N$, that is, a sequence of nilmanifolds $N_i/N_{i-1}$ acting on $N/N_{i-1}$ by left translation with orbit space $N/N_i$. Also $D_i$ in the upper central series of $D$ is just $N_i \cap D$ and $N_i/\pi_i$ acting on $N/N_{i-1}$ is isomorphic to $(N_i/N_{i-1})/(D_i/D_{i-1})$. We recall (Theorem 2 of [7]) that $D_i/D_{i-1}$ is free abelian, a fact which is not true in general of the factor groups of the lower central series of $D$. Thus $M$ is expressed as a sequence of extensions by tori whose fundamental groups are $D_i/D_{i-1}$.

Let $f$ be a homeomorphism of $M$ and let $f_*$ be the automorphism it induces on the fundamental group $D$. Since we have not mentioned base points yet $f_*$ is only defined up to an inner automorphism of $D$ but that is sufficient for our purposes. $f_*$ preserves the upper central series of $D$ and so induces automorphisms $\varphi_i: D_i/D_{i-1} \to D_i/D_{i-1}$ for $i = 1, \cdots, c$. We shall prove

THEOREM. If $f$ is an Anosov diffeomorphism then none of the $\varphi_i$'s have a root of unity as an eigenvalue.
In [5] Hirsch proved this for the map induced by \( f \) on \( H_1(M; R) \). Our proof uses a spectral sequence to calculate the Lefschetz number of \( f \) and shows the remarkable fact that it is independent of the twists with which the tori are put together to make up \( M \).

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The theorem for \( f \) is equivalent to the theorem for a power of \( f \) so (using the fact that \( f \) has periodic points, see Proposition 1.7 of [2]) we may assume \( f \) has a fixed point. Again we may assume this fixed point is the base point \( eD \) of \( M \) by conjugating \( f \) by a translation since \( N \) acts transitively on \( M \).

By Theorem 5 of [6], \( f_* : D \to D \) extends uniquely to an automorphism \( G : N \to N \). Since \( G \) preserves the subgroup \( D \) it induces a diffeomorphism \( g \) of the homogeneous space \( N/D = M \). The diffeomorphisms \( f, g \) of \( M \) induce the same automorphism of the fundamental group \( D \) and so, since \( M \) is a \( K(D, 1) \) (because its universal cover \( N \) is contractible, see [10, p. 180]), are freely homotopic, see [12, Theorem 8.1.11], and induce the same map of \( H_* (M) \). Therefore \( L(f) = L(g) \).

The automorphism \( \varphi_i \) of the fundamental group of the \( i \)th torus of \( M \) is induced by an automorphism \( g_i \) say of this torus and \( g : M \to M \) is the extension of \( g_1 \) by \( g_2 \) by \( \cdots \) by \( g_c \). We show that \( L(g) = L(g_1 \times \cdots \times g_c) \).

A special case of this was noticed by Bowen [1, p. 395]. In fact it follows from the next lemma by induction on \( c \) and the observation that the condition about trivial action is satisfied because the series \( \{ D_i \} \) is central.

**Lemma.** Let \( \pi : X, \star \to B, \star \) be a fibre bundle with fibre \( F = \pi^{-1} \star \) and suppose \( \pi_1(B) \) acts trivially on the homology of \( F \). Assume that at least one of \( B, F \) is compact. Let \( (\psi, \chi) \) be a bundle map, i.e. a pair of continuous maps \( s.t. \) the diagram

\[
\begin{array}{ccc}
X, \star & \longrightarrow & X, \star \\
\downarrow \pi & & \downarrow \pi \\
B, \star & \longrightarrow & B, \star \\
\end{array}
\]

commutes and let \( \omega = \psi|F \). Then \( L(\psi) = L(\chi \times \omega) \).

**Remark.** \( \psi : X \to X \) and \( \chi \times \omega : B \times F \to B \times F \) differ by twists in the fibres so the lemma says that the Lefschetz number ignores these twists. If \( \psi = \text{id}_X \) then the result reduces to the multiplicative property of the Euler characteristic (Theorem 9.3.1 of [12]) which however is true without the condition of trivial action. This condition is required here since the
Klein bottle $K$ is an $S^1$ bundle over $S^1$ failing to satisfy it and the map
$\varphi: K \to K$ that induces the identity in the fibre but wraps the base three times round itself has Lefschetz number $-2$ but the corresponding map of $T^2$ has Lefschetz number 0.

**Proof.** We use cubical singular homology with real coefficients and the Serre spectral sequence, see [9] and [4]. Let $0^n(X)$ be the real vector space with basis all maps of the standard $n$-cube $I^n$ into $X$ such that all vertices are mapped to $\ast$. Filter $0^n(X)$ as follows. Take a basis element $\sigma \in 0^n(X)$, $\sigma: I^n \to X$ and define $p$ to be the least integer such that $\pi_0(u_1, \ldots, u_n)$ is independent of $u_{p+1}, \ldots, u_n$. Then $\sigma \in 0^n_p(X)$. Now $\varphi: X, \ast \to X, \ast$ induces a chain map of $0^n_p(X)$ to itself which preserves the filtration by $p$. So $\varphi$ induces a map which we denote by $\varphi_\ast$ on every term $E^r_p$ of the spectral sequence obtained from $0^n_p$.

Define

$$L(\varphi, E^r) = \sum_{p, q} (-1)^{p+q} \text{trace}(\varphi_\ast: E^p_{pq} \to E^p_{pq}).$$

Then $L(\varphi, E^r) = L(\varphi, E^{r+1})$ by a version of the Hopf trace theorem, see e.g. 5.1.18 of [4].

Now $E^p_{pq} = H_p(B; H_q(F)) = H_p(B) \otimes H_q(F)$ by the assumption of trivial action and $H_n(B \times F) = \bigoplus_{p+q=n} H_p(B) \otimes H_q(F)$ by the Künneth formula. So $L(\varphi, E^2) = L(\chi \times \omega)$.

Since one of $B$, $F$ is compact there is an $m$ such that $E^{m}_{pq} = E^\infty_{pq}$ and

$$L(\varphi, E^\infty) = \sum_{p, q} (-1)^{p+q} \text{trace}(\varphi_\ast: E^\infty_{pq} \to E^\infty_{pq})$$

$$= \sum_{n} (-1)^n \text{trace}(\varphi_\ast: H_n(X) \to H_n(X)) = L(\varphi).$$

Therefore $L(\varphi) = L(\varphi, E^\infty) = L(\varphi, E^m) = L(\varphi, E^2) = L(\chi \times \omega)$.

**Proof of Theorem.** Now we can calculate

$$L(f) = L(g) = L(g_1 \times \cdots \times g_o) = \prod (1 - \lambda)$$

where the product is taken over all eigenvalues $\lambda$ counted with multiplicity of all the maps $g_i$ [11, p. 769]. If one of these eigenvalues is a $j$th root of unity then $L(f^j) = 0$ according to this calculation. But $L(f^j) \neq 0$ if we can show that all the fixed points of $f^j$ have the same Lefschetz index. (Recall that $f$ has a fixed point.) This is easy if the expanding bundle $E^u$ is orientable, see [3, p. 123]. Moreover if $E^u$ is not orientable we can use the same trick as Franks. Namely we construct a covering $f'$ of $f$ on the covering space of $M$ corresponding to that subgroup $H$ of $\pi_1(M)$ which is the inverse image of $2D/[D, D]$ under the Hurewicz map $\pi_1(M) = D \to D/[D, D] = H_1(M; Z)$. Then $f'$ is an Anosov diffeomorphism.
with orientable expanding bundle so the map induced by \( f' \) on \( H \) and hence the map induced by \( f \) on \( D \) has no eigenvalues which are roots of unity. This completes the proof of the theorem.

**Remark.** Suppose now \( g \) is a hyperbolic nilmanifold automorphism. We see from the calculation above that the zeta function \( \zeta(g) \) is the same as \( \zeta(g_1 \times \cdots \times g_n) \) and this is obtained \([11, p. 769]\) from the false zeta function \( \zeta(t) = \prod (1 - \lambda_i \lambda_{i_2} \cdots \lambda_{i_k} t)^{-1} \) where the product is taken over all \( (i_1, \cdots, i_k) \) s.t. \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \). Now \( g \) induces an automorphism of the Lie algebra of \( N \) and if this Lie algebra is not abelian then there must be eigenspaces corresponding to eigenvalues \( \lambda_i, \lambda_j \) say whose bracket is not zero making \( \lambda_i \lambda_j \) an eigenvalue too. So the zeta function above of a toral automorphism can only be the zeta function of a nontoral nilmanifold automorphism if a factor for which \( k = 1 \) cancels with a factor for which \( k = 2 \).

**References**


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