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ON FINITE INVARIANT MEASURES FOR MARKOV OPERATORS[1]

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ABSTRACT. Two lemmas on proper vectors of convex linear combination of operators and semigroups in a Banach space are proved. They are applied to problems of invariant measures for Markov operators.

1. Proper vectors of convex linear combinations.

LEMMA 1. *Let $\{P_i\}$ be commuting operators on the Banach space B with $\|P_i\|\leqq 1$. Let $P=\sum_{i=1}^{\infty}\alpha_iP_i$ where $\alpha_i>0$, $\sum\alpha_i=1$. If $Px=\lambda x$, $|\lambda|=1$, then $P_ix=\lambda x$, $i=1,2,\cdots$.*

PROOF. Fix i_0; then

$$P=\alpha_{i_0}P_{i_0}+\sum_{i\neq i_0}\alpha_iP_i=\alpha_{i_0}P_{i_0}+(1-\alpha_{i_0})Q$$

where $Q=\sum_{i\neq i_0}(\alpha_i/(1-\alpha_{i_0}))P_i$; necessarily $\|Q\|\leqq 1$. By a lemma of Foguel [1, Lemma 2.1], $\|(P_{i_0}-Q)P^n\|\to_{n\to\infty}0$. But

$$\|(P_{i_0}-Q)P^nx\|=\|\lambda^n(P_{i_0}x-Qx)\|=\|P_{i_0}x-Qx\|,$$

hence $P_{i_0}x=Qx$ and necessarily $P_{i_0}x=\lambda x$.

REMARK. The condition $\alpha_i>0$ is not essential in the lemma. If the α_i are nonzero and $\sum_{i=1}^{\infty}|\alpha_i|=|\lambda|$, the conclusion remains true, with $P_ix=(\operatorname{sgn}\lambda/\operatorname{sgn}\alpha_i)x$. To see that, consider

$$P=\sum_{i=1}^{\infty}\frac{|\alpha_i|}{|\lambda|}\left(\frac{\alpha_i}{|\alpha_i|}P_i\right).$$

Let us consider a strongly continuous semigroup of operators on B, P_t, with $\|P_t\|\leqq 1$. Given a measurable, nonnegative function $\phi(t)$, $t\geqq 0$, with $\int^{\infty}\phi(t)\,dt=1$, define $R_\phi=\int_0^{\infty}\phi(t)P_t\,dt$; extend $\phi(t)$ to be zero for

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$t<0$. For every $x \in B$, the function $\phi(t)P_t x$ is strongly measurable; see the proof of Theorem 9.2.2 in [5]. Since $\int_0^\infty \|\phi(t)P_t x\|\, dt<\infty$, it is Bochner integrable, and $\|R_\phi\| \leqq \int_0^\infty \phi(t)\, dt$ [5, Theorem 3.5.2].

We wish to find $R_\phi R_\psi$:

$$\begin{aligned}\langle R_\phi R_\psi x, x^*\rangle &= \int_0^\infty \phi(t)\langle P_t R_\psi x, x^*\rangle\, dt\\ &= \int_0^\infty \phi(t)\langle R_\psi x, P_t^* x^*\rangle\, dt\\ &= \int_0^\infty \phi(t)\left(\int_0^\infty \psi(s)\langle P_s x, P_t^* x^*\rangle\, ds\right) dt\\ &= \int_0^\infty \phi(t)\left(\int_0^\infty \psi(s)\langle P_{s+t} x, x^*\rangle\, ds\right) dt.\end{aligned}$$

Changing variables, one obtains

$$\begin{aligned}\langle R_\phi R_\psi x, x^*\rangle &= \int_0^\infty \phi(t)\left(\int_t^\infty \psi(r-t)\langle P_r x, x^*\rangle\, dr\right) dt\\ &= \int_0^\infty \phi(t)\left(\int_0^\infty \psi(r-t)\langle P_r x, x^*\rangle\, dr\right) dt\\ &= \int_0^\infty \left(\int_0^\infty \phi(t)\psi(r-t)\, dt\right)\langle P_r x, x^*\rangle\, dr\\ &= \int_0^\infty (\phi * \psi)(r)\langle P_r x, x^*\rangle\, dr\end{aligned}$$

since Fubini's theorem certainly applies. Thus $R_\phi R_\psi = R_{\phi*\psi}$.

LEMMA 2. *Let $R_\phi = \int_0^\infty \phi(t)P_t\, dt$ be the operator defined above. If $R_\phi x = \lambda x$, $|\lambda|=1$, then $\lambda=1$ and $P_t x = x$, $t \geqq 0$.*

PROOF. Suppose first that ϕ majorizes a positive multiple of the characteristic function of a certain interval. That is, there exist $c>0$ and $0 \leqq a<b$ such that $\phi \geqq c1_{[a,b]}$. We choose c small enough so that $c(b-a)<1$. Denote $1_{[a,b]}$ by χ; then

$$\begin{aligned}R_\phi &= cR_\chi + R_{\phi-c\chi}\\ &= c(b-a)(R_\chi/(b-a)) + (1-c(b-a))(R_{\phi-c\chi}/(1-c(b-a))).\end{aligned}$$

Since $\|R_\chi/(b-a)\|$, $\|R_{\phi-c\chi}/(1-c(b-a))\| \leqq 1$, the former lemma can be applied to get

$$\frac{1}{b-a}\int_a^b P_t x\, dt = \lambda x.$$

Now let s_0, $a<s_0<b$ and $\varepsilon>0$ be given. Let $\delta>0$ be such that $|s-s_0|<\delta \Rightarrow \|P_s x-P_{s_0}x\|<\varepsilon$. If s is also in the interval (s_0, b), a repetition of the argument above shows

$$\frac{1}{s-s_0}\int_{s_0}^{s} P_t x\,dt = \lambda x.$$

But

$$\left\|\frac{1}{s-s_0}\int_{s_0}^{s} P_t x\,dt - P_{s_0}x\right\| < \varepsilon \quad \text{for} \quad |s-s_0|<\delta.$$

Since ε is arbitrary, $P_{s_0}x=\lambda x$. Thus $P_t x=\lambda x$ for all $a<t<b$. Now, for any $t>0$, let n be so large that $t/n<b-a$; then, for a certain positive integer k, $a<kt/n<(k+1)t/n<b$. Hence

$$\lambda x = P_{t/n}^{k+1}x = P_{t/n}P_{t/n}^{k}x = \lambda P_{t/n}x \quad \text{and therefore} \quad P_t x = x.$$

Necessarily $\lambda=1$: $\lambda x=P_{2t}x=P_t^2 x=\lambda^2 x$.

For the case of a general ϕ, we choose $0\leqq\psi\leqq\phi$ bounded, so that $\psi * \psi$ is continuous (convolution of a L_1 function with a L_∞ function; see [7, Theorem, p. 4]). Let

$$\psi_1(t) = \left(\int_0^\infty \psi(s)\,ds\right)^{-1}\psi(t).$$

Then $R_{\psi_1}x=\lambda x$, implying $R_{\psi_1 * \psi_1}x=\lambda^2 x$ and by the previous part, $P_t x=x$ for all $t\geqq 0$ and $\lambda^2=1$. But then $\lambda x=R_\phi x=x$ and necessarily $\lambda=1$.

2. **Application to Markov operators.** Let (x, Σ, m) be a finite measure space. We shall use the notation and definitions of [**2**].

Applied to Markov operators and invariant measures, Lemma 1 reads:

THEOREM 1. *Let $P=\sum_{i=1}^\infty \alpha_i P_i$ where P_i are commuting Markov operators, $\alpha_i>0$ and $\sum \alpha_i=1$. Then a finite invariant measure for P is invariant for all P_i.*

REMARK. Suppose P, Q are Markov operators, $P1\leqq Q1$ and P dominates Q in the following sense: $P\geqq\alpha Q$ for some $0<\alpha<1$. Then $P=\alpha Q+(1-\alpha)(P-\alpha Q)/(1-\alpha)$ is a convex linear combination of Markov operators: clearly $(P-\alpha Q)/(1-\alpha)$ is positive and $((P-\alpha Q)/(1-\alpha))1\leqq Q1$ implies it is a contraction.

The following two results are known. Corollary 1 is due to S. Horowitz [**4**] (his result is slightly more general), and Corollary 2 to A. Brunel (unpublished). Let us show how to derive them from Theorem 1.

COROLLARY 1. *Let Π be a commutative semigroup of Markov operators having no finite invariant measure equivalent to m. Then there exist $P_i \in \Pi$ and $\alpha_i>0$, $\sum \alpha_i=1$, such that $\sum_{i=1}^\infty \alpha_i P_i$ is not conservative.*

PROOF. M. Lin has shown in [3] that $\inf_{P\in\Pi} mP(A)>0$ for every $A\in\Sigma$, $m(A)>0$, is a necessary and sufficient condition for a finite equivalent invariant measure for Π. Thus there exist a sequence P_i such that there is no finite equivalent invariant measure common to all P_i. By Theorem 1 neither does any $Q=\sum_{i=1}^{\infty}\alpha_i P_i$ with $\alpha_i>0$, $\sum\alpha_i=1$, have such a measure. Brunel's result in [6] then supplies an operator $\sum_{j=0}^{\infty}\beta_j Q^j$, $\beta_j\geqq 0$, $\sum\beta_j=1$, which is not conservative. From the condition for conservativity in [8], $Ph\leqq h$ for $0\leqq h\leqq 1\Rightarrow Ph=h$, neither is $(1/(1-\beta_0))\sum_{j=1}^{\infty}\beta_j Q^j$, which is clearly a convex linear combination of members of Π.

COROLLARY 2. *Let P be a Markov operator and $Q=\sum_{i=0}^{\infty}\alpha_i P^i$ where $\alpha_i\geqq 0$, $\sum\alpha_i=1$. Then an invariant measure u for Q is invariant for P^r, where r is the greatest common divisor of $n>0$ such that $\alpha_n>0$.*

PROOF. There exist $n_1,\cdots,n_k$ with $\alpha_{n_j}>0$ and nonzero integers $q_1,\cdots,q_k$ such that $r=\sum_{j=1}^{k}q_jn_j$. Write $\sum_1 q_jn_j$ for the summation over q_j positive and $\sum_2 q_jn_j$ for the summation over q_j negative. Since, by Theorem 1, $uP^{n_j}=u$, $j=1,\cdots,k$, we have

$$uP^r=(uP^{-\Sigma_2 q_jn_j})P^r=uP^{\Sigma_1 q_jn_j}=u.$$

Let $\{P_t\}$ be a strongly continuous semigroup of Markov operators. Then Lemma 2 reads as follows:

THEOREM 2. *A finite measure is invariant for $\{P_t\}$ if and only if it is invariant for any operator*

$$\int_0^\infty \phi(t)P_t\,dt,\quad \textit{where } \phi(t)\geqq 0,\quad \int_0^\infty \phi(t)\,dt=1.$$

Using Brunel's result in [6] we may conclude:

COROLLARY. *If $\{P_t\}$ has no m-equivalent finite invariant measure, then there exists a function $\phi(t)$ with $\phi(t)\geqq 0$, $\int_0^\infty\phi(t)\,dt=1$, such that $\int_0^\infty\phi(t)P_t\,dt$ is not conservative.*

Indeed, for any $R_\psi=\int_0^\infty\psi(t)P_t\,dt$, if it is conservative, there are $\alpha_n\geqq 0$, $\sum\alpha_n=1$, such that $\sum\alpha_nR_\psi^n$ is not conservative. Put $\phi(t)=\sum\alpha_n(\phi^*)^n$.

ADDED IN PROOF. Lemma 2 (and Theorem 2) hold for the general case of a strongly continuous operator representation by operators of norm 1, of a locally compact connected and metrizable Abelian group. Proofs are virtually the same, with necessary modifications.

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