CONTINUOUS DEPENDENCE OF FIXED POINT SETS

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Abstract. The stability of the fixed point sets of a uniformly convergent sequence of set valued contractions is proved under the assumption that the maps are defined on a closed bounded subset $B$ of Hilbert space and take values in the family of nonempty closed convex subsets of $B$.

In [1] the convergence of a sequence of fixed points of a convergent sequence of set valued contractions was investigated in a metric space setting. By restricting the underlying space to be a Hilbert space we prove the convergence of the sequence of fixed point sets of a convergent sequence of set valued contractions. This also extends a similar result for point valued maps [2, Theorem (10.1.1)] to the set valued case.

Let $A$ be a closed bounded subset of a Hilbert space $H$, $d$ the norm of $H$, and $D$ the Hausdorff metric on the closed subsets of $A$ generated by $d$. We assume that the family of set valued maps $F_k$, $k=0, 1, \cdots$, satisfy

1. $F_k(x)$ is a nonempty closed convex subset of $A$ for each $x \in A$.
2. Each $F_k$ is a set valued contraction, i.e. there is a $K \in [0, 1)$ such that $D(F_k(x), F_k(y)) \leq Kd(x, y)$ for $x, y \in A$ and $k=0, 1, \cdots$. 
3. $\lim_{k \to \infty} D(F_k(x), F_0(x)) = 0$ uniformly for all $x \in A$.

Theorem. If the conditions 1-3 are satisfied then the fixed point sets of the sequence $\{F_k\}$, $k=1, 2, \cdots$, converge to the fixed point set of $F_0$ in the Hausdorff metric $D$.

Before proving the theorem some lemmas on the closest point projection map associated with each $F_k$ are required. For $k=0, 1, \cdots$, define the maps $f_k$ by $f_k(x)$ = \{unique closest point in $F_k(x)$ to $x$} for $x \in A$. The iterates of each $f_k$ are denoted by $f_k^n$, $n=2, 3, \cdots$. The distance between any $x \in A$ and closed subset $C$ of $A$ will be $d(x, C) = \inf_{c \in C} d(x, c)$.

The following result was given in [3, Lemma 5] for a finite dimensional space, but the statement and proof are valid for any Hilbert space.
Lemma 1. If $E$ and $F$ are closed convex subsets of $H$ and $e$ and $f$ are the closest points in $E$ and $F$ to a point $v \in H$, then
\[
d(e,f) \leq (h^2 + 4hl)^{1/2}, \quad \text{where } h = D(E,F) \text{ and } l = d(v,E).
\]

For $s \geq 0$ we define the continuous monotone increasing function $g(s) = (s^2 + 4sr)^{1/2}$, where $r$ is the diameter of $A$.

Lemma 2. The maps $f_k$, $k = 0, 1, \cdots$, are equicontinuous on $A$.

Proof. For any $k = 0, 1, \cdots$, and $x, y \in A$ let $q$ denote the closest point in $F_k(x)$ to $y$. Consider the inequality
\[
d(f_k(x), f_k(y)) \leq d(f_k(x), q) + d(q, f_k(y)).
\]
The term $d(f_k(x), q)$ is bounded by $d(x, y)$, since projection onto a closed convex set is nonexpansive. Lemma 1 implies that $d(q, f_k(y))$ is bounded by $g(D(F_k(x), F_k(y)))$. Since $F_k$ is a set valued contraction and $g$ is monotone we have $g(D(F_k(x), F_k(y))) \leq g(Kd(x, y))$. The inequality (1) can then be written as $d(f_k(x), f_k(y)) \leq d(x, y) + g(Kd(x, y))$. The map $g$ is continuous and $g(s) \to 0$ as $s \to 0$. Therefore, the latter inequality proves the lemma.

Lemma 3. The sequence of maps $\{f_k^n\}$, $k = 1, 2, \cdots$, converges uniformly on $A$ to $f_0^n$, for each $n$.

Proof. For $n = 1$ the uniform convergence follows from,
\[
d(f_k(x), f_0(x)) \leq g(D(F_k(x), F_0(x)))
\]
and the uniform convergence of the maps $F_k$ to $F_0$. Make the induction assumption that $f_k^{n-1}, k = 1, 2, \cdots$, converges uniformly on $A$ to $f_0^{n-1}$. Given $\varepsilon > 0$ there is a $\delta > 0$ such that $u, v \in A$ and $d(u, v) < \delta$ implies $d(f_k(u), f_k(v)) < \varepsilon/2$, for all $k$, in view of the equicontinuity of the $f_k$. The uniform convergence of the sequences $\{f_k\}$ and $\{f_k^{n-1}\}$ to $f_0$ and $f_0^{n-1}$ permits the choice of an integer $N$ such that $k \geq N$ implies
\[
d(f_k^{n-1}(x), f_0^{n-1}(x)) < \delta \quad \text{and} \quad d(f_k(x), f_0(x)) < \varepsilon/2
\]
for all $x \in A$. Considering the inequality
\[
d(f_k^n(x), f_0^n(x)) \leq d(f_k(f_k^{n-1}(x)), f_k^n(f_0^{n-1}(x))) + d(f_k^n(f_0^{n-1}(x)), f_0^n(f_0^{n-1}(x)))
\]
the remarks above imply that for $k \geq N$ the right side of the inequality is strictly less than $\varepsilon$. This proves uniform convergence for all $n$.

Proof of the theorem. By a result of Nadler [1, Theorem 5] the sequence of iterates $\{f_k^n(x)\}$ converges to a fixed point of $F_k$ for $k = 0, 1, \cdots$, and all $x \in A$. If $P_k$ denotes the fixed point set of $f_k$, then this same
result contains the estimate

$$(2) \quad d(f_k^n(x), P_k) \leq \sum_{i=n}^{\infty} (r + i)K^i. $$

Each $P_k$ is a closed subset and can be written as

$$P_k = \left\{ y \in A : \lim_{n \to \infty} f_k^n(x) = y, \ x \in A \right\}. $$

Given $\epsilon > 0$ choose any $x \in A$ and let $P_k(x) = \lim_{n \to \infty} f_k^n(x), \ k = 0, 1, \cdots$. Consider the inequality

$$(3) \quad d(P_k(x), P_0(x)) \leq d(P_k(x), f_k^n(x)) + d(f_k^n(x), f_0^n(x)) + d(f_0^n(x), P_0(x)). $$

By the estimate (2) we can choose an integer $N$ such that $d(f_k^N(x), P_k(x)) < \epsilon/3$ for $k = 0, 1, \cdots$, and all $x \in A$. The uniform convergence of $\{f_k^N\}$ to $f_0^N$ permits the choice of an integer $M$ such that $k \geq M$ implies $d(f_k^N(x), f_0^N(x)) < \epsilon/3$ for all $x \in A$. Therefore, by (3), $d(P_k(x), P_0(x)) < \epsilon$ for all $x \in A$. Since the points $P_k(x)$ range over $P_k$ as $x$ ranges over $A$, we have shown that $D(P_k, P_0) < \epsilon$ for $k \geq M$. This proves convergence of $P_k$ to $P_0$ in the $D$ metric.

REFERENCES


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