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THE COMPACTIFICATIONS TO WHICH AN ELEMENT OF $C^*(X)$ EXTENDS

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ABSTRACT. We first determine a necessary and sufficient condition for a function $f \in C^*(X)$, which extends to a compactification of X, to extend to a smaller compactification. We apply this result to show that when $|\beta X \backslash X| \leqq \aleph_0$ there is an $f \in C^*(X)$ which extends to no compactification other than βX. Two examples show that when $\aleph_0 < |\beta X \backslash X| \leqq c$ no such definite result may be obtained.

One property which distinguishes the Stone-Čech compactification βX as distinct from all other Hausdorff compactifications of the completely regular space X is that each $f \in C^*(X)$ has an extension $f^\beta \in C^*(\beta X)$. Another way of expressing this is that $\beta X \in K(f)$ for each $f \in C^*(X)$, where $K(f)$ denotes the set of Hausdorff compactifications of X to which f has an extension. This needs some clarification. A *compactification* of X is a compact Hausdorff space αX containing X as a dense subspace. $\alpha_1 X$ and $\alpha_2 X$ are *equivalent* compactifications of X if there is a homeomorphism $h:\alpha_1 X \to \alpha_2 X$ such that h restricted to $X \subseteq \alpha_1 X$ is the identity mapping onto $X \subseteq \alpha_2 X$. Generally, we will not distinguish between equivalent compactifications. By $K(f)$ we mean, then, a set of compactifications of X having the properties (i) if $\alpha X \in K(f)$ then there is an $f^\alpha \in C^*(\alpha X)$ such that $f^\alpha|_X = f$ (f has an *extension* f^α to αX) and (ii) if γX is any compactification to which f has an extension then there is an element of $K(f)$ equivalent to γX. When we say $\beta X \in K(f)$ we mean that there is in $K(f)$ a compactification from the class of compactifications equivalent to some specific construction of βX, say that in [**1**, Chapter 6].

For compactifications $\alpha_1 X$ and $\alpha_2 X$ we say that $\alpha_1 X \geqq \alpha_2 X$ if there is a mapping $h:\alpha_1 X \to \alpha_2 X$ such that $h|_X = \mathrm{id}$. If $\alpha_1 X \geqq \alpha_2 X$ and they are *not* equivalent we write $\alpha_1 X > \alpha_2 X$. Our first result is a necessary and sufficient condition that a given element of $K(f)$ will be minimal with respect to this order.

THEOREM 1. *αX is a minimal element of $K(f)$ if and only if f^α is* 1-1 *on $\alpha X \backslash X$.*

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PROOF. If $f^\alpha(x_0)=f^\alpha(x_1)$ for some $x_0, x_1 \in \alpha X\backslash X$ then we may obtain a compactification γX for which $\alpha X>\gamma X$ by taking the quotient space of αX obtained by identifying x_0 and x_1. γX is equivalent to some $\delta X \in K(f)$ and $\alpha X>\delta X$.

Conversely, suppose there were a $\gamma X\in K(f)$ for which $\alpha X \geqq \gamma X$. If $h:\alpha X\to\gamma X$ is the identity map on $X\subseteq\alpha X$ then $f^\gamma\circ h=f^\alpha$ so that if f^α is 1-1 on $\alpha X\backslash X$ then h is 1-1 on $\alpha X\backslash X$. Since h restricted to the dense set $X\subseteq\alpha X$ is a homeomorphism onto the dense set $X\subseteq\gamma X$ we have that $h(\alpha X\backslash X)\subseteq \gamma X\backslash X$ [**1**, Lemma 6.11]. Thus, h is 1-1 from the compact space αX onto the Hausdorff space γX and is, therefore, a homeomorphism. αX and γX are equivalent and since both are members of $K(f)$ we have that $\alpha X=\gamma X$.

COROLLARY. *If X is realcompact and not compact then for no $f\in C^*(X)$ is $K(f)=\{\beta X\}$.*

PROOF. $|\beta X\backslash X|\geqq 2^c$ [**1**, p. 136].

Simple cardinality considerations (as above) dictate that if $K(f)=\{\beta X\}$ then $|\beta X\backslash X|\leqq c$. We will next show that if $|\beta X\backslash X|\leqq\aleph_0$ then there is always an $f\in C^*(X)$ for which $K(f)=\{\beta X\}$.

LEMMA. *If Y is a completely regular space and $\{y_k\}$ is a sequence of distinct points in Y then for each $n\geqq 1$ there is an $f_n\in C^*(Y)$ such that*

(a) $f_n(Y)\subseteq[0, 1]$,

(b) $f_n(y_p)=0,\ 1\leqq p<n$,

(c) $f_n(y_n)=1$,

(d) $f_n(y_p)$ *is rational if* $p>n$.

PROOF. By hypothesis there is a continuous $g_0: Y\to[0, 1]$ such that $g_0(y_p)=0$, $1\leqq p<n$, and $g_0(y_n)=1$. We define a sequence $\{g_k\}$ as follows. If $g_0, g_1, \cdots, g_{k-1}$ have been defined and $g_{k-1}(y_{n+k})=c$ then if $c\in\{0, 1/2^{k-1}, 2/2^{k-1}, \cdots, (2^{k-1}-1)/2^{k-1}, 1\}$ define $g_k=g_{k-1}$. Otherwise let $g_k=h_k\circ g_{k-1}$ where, if $c\in(p/2^{k-1}, p+1/2^{k-1})$, then h_k is the homeomorphism of $[0, 1]$ onto itself which is the identity mapping on $[0, p/2^{k-1}]\cup[p+1/2^{k-1}, 1]$, sends $[p/2^{k-1}, c]$ onto $[p/2^{k-1}, 2p+1/2^k]$, and sends $[c, p+1/2^{k-1}]$ onto $[2p+1/2^k, p+1/2^{k-1}]$. $\{g_k\}$ is a uniformly convergent sequence since $|g_k(x)-g_{k+p}(x)|\leqq 1/2^k+1/2^{k+1}+\cdots+1/2^{k+p}<1/2^{k-1}$. Thus, $f_n=\lim g_k$ is continuous and clearly has the desired properties.

THEOREM 2. *If $|\beta X\backslash X|\leqq\aleph_0$ then there is an $f\in C^*(X)$ such that $K(f)=\{\beta X\}$.*

PROOF. Let $\beta X\backslash X=\{x_1, x_2, \cdots\}$ and let $\{f_n\}$ be the sequence guaranteed by the lemma. Define $g:\beta X\to R^1$ by $g(x)=\sum_n (\pi/4)^n f_n(x)$. Then $g\in C^*(\beta X)$

and if $f=g|_X$ it follows that $f^\beta=g$. If $g(x_p)=g(x_q)$ then $\sum_1^p (\pi/4)^n f_n(x_p)=\sum_1^q (\pi/4)^n f_n(x_q)$, contradicting the transcendence of π ($f_n(x_p), f_n(x_q)$ are all rational). Thus, f^β is 1-1 on $\beta X\setminus X$ so that, by Theorem 1, $K(f)=\{\beta X\}$.

If $\aleph_0<|\beta X\setminus X|\leqq c$, we cannot obtain such a definitive result as can be seen by the following two examples.

EXAMPLE 1. Let $X=W(\omega_1+1)\times W(\omega_1)$. Then $\beta X\setminus X$ is homeomorphic to $W(\omega_1+1)$ [**1**, pp. 137–138] so that no f^β can be 1-1 $\beta X\setminus X$. Thus $K(f)\neq\{\beta X\}$ for any $f\in C^*(X)$. In this case $|\beta X\setminus X|=\aleph_1\leqq c$.

EXAMPLE 2. Let $X=[0, 1]\times W(\omega_\alpha)$ where α is a nonlimit ordinal for which $\aleph_\alpha>c$. Then $\beta X=[0, 1]\times W(\omega_\alpha+1)$ so that $\beta X\setminus X$ is homeomorphic to $[0, 1]$. Let $f:X\rightarrow R$ be the projection onto the first coordinate. Clearly $f^\beta|_{\beta X\setminus X}=\mathrm{id}_{[0,1]}$ so that $K(f)=\{\beta X\}$.

REFERENCE

1. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

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