THE COMPACTIFICATIONS TO WHICH
AN ELEMENT OF $C^*(X)$ EXTENDS

RICHARD E. CHANDLER AND RALPH GELLAR

Abstract. We first determine a necessary and sufficient condition for a function $f \in C^*(X)$, which extends to a compactification of $X$, to extend to a smaller compactification. We apply this result to show that when $|\beta X \setminus X| \leq \kappa_0$, there is an $f \in C^*(X)$ which extends to no compactification other than $\beta X$. Two examples show that when $\kappa_0 < |\beta X \setminus X| \leq c$ no such definite result may be obtained.

One property which distinguishes the Stone-Čech compactification $\beta X$ as distinct from all other Hausdorff compactifications of the completely regular space $X$ is that each $f \in C^*(X)$ has an extension $f^\beta \in C^*(\beta X)$. Another way of expressing this is that $\beta X \in K(f)$ for each $f \in C^*(X)$, where $K(f)$ denotes the set of Hausdorff compactifications of $X$ to which $f$ has an extension. This needs some clarification. A compactification of $X$ is a compact Hausdorff space $\alpha X$ containing $X$ as a dense subspace. $\alpha_1 X$ and $\alpha_2 X$ are equivalent compactifications of $X$ if there is a homeomorphism $h: \alpha_1 X \to \alpha_2 X$ such that $h$ restricted to $X \subseteq \alpha_1 X$ is the identity mapping onto $X \subseteq \alpha_2 X$. Generally, we will not distinguish between equivalent compactifications. By $K(f)$ we mean, then, a set of compactifications of $X$ having the properties (i) if $\alpha X \in K(f)$ then there is an $f^\alpha \in C^*(\alpha X)$ such that $f^\alpha|_X = f$ has an extension $f^\alpha$ to $\alpha X$ and (ii) if $\gamma X$ is any compactification to which $f$ has an extension then there is an element of $K(f)$ equivalent to $\gamma X$. When we say $\beta X \in K(f)$ we mean that there is in $K(f)$ a compactification from the class of compactifications equivalent to some specific construction of $\beta X$, say that in [1, Chapter 6].

For compactifications $\alpha_1 X$ and $\alpha_2 X$ we say that $\alpha_1 X \geq \alpha_2 X$ if there is a mapping $h: \alpha_1 X \to \alpha_2 X$ such that $h|_X = \text{id}$. If $\alpha_1 X \geq \alpha_2 X$ and they are not equivalent we write $\alpha_1 X > \alpha_2 X$. Our first result is a necessary and sufficient condition that a given element of $K(f)$ will be minimal with respect to this order.

Theorem 1. $\alpha X$ is a minimal element of $K(f)$ if and only if $f^\alpha$ is 1-1 on $\alpha X \setminus X$.
Proof. If \( f^*(x_0) = f^*(x_1) \) for some \( x_0, x_1 \in \alpha X \setminus X \) then we may obtain a compactification \( \gamma X \) for which \( \alpha X \geq \gamma X \) by taking the quotient space of \( \alpha X \) obtained by identifying \( x_0 \) and \( x_1 \). \( \gamma X \) is equivalent to some \( \delta X \in K(f) \) and \( \alpha X \geq \delta X \).

Conversely, suppose there were a \( \gamma X \in K(f) \) for which \( \alpha X \geq \gamma X \). If \( h: \alpha X \rightarrow \gamma X \) is the identity map on \( X \subseteq \alpha X \) then \( f^* \circ h = f^* \) so that if \( f^* \) is 1-1 on \( \alpha X \setminus X \) then \( h \) is 1-1 on \( \alpha X \setminus X \). Since \( h \) restricted to the dense set \( X \subseteq \alpha X \) is a homeomorphism onto the dense set \( X \subseteq \gamma X \) we have that \( h(\alpha X \setminus X) \subseteq \gamma X \setminus X \) [1, Lemma 6.11]. Thus, \( h \) is 1-1 from the compact space \( \alpha X \) onto the Hausdorff space \( \gamma X \) and is, therefore, a homeomorphism. \( \alpha X \) and \( \gamma X \) are equivalent and since both are members of \( K(f) \) we have that \( \alpha X = \gamma X \).

Corollary. If \( X \) is realcompact and not compact then for no \( f \in C^*(X) \) is \( K(f) = \{ \beta X \} \).

Proof. \( |\beta X \setminus X| \geq 2^c \) [1, p. 136].

Simple cardinality considerations (as above) dictate that if \( K(f) = \{ \beta X \} \) then \( |\beta X \setminus X| \leq c \). We will next show that if \( |\beta X \setminus X| \leq \kappa_0 \) then there is always an \( f \in C^*(X) \) for which \( K(f) = \{ \beta X \} \).

Lemma. If \( Y \) is a completely regular space and \( \{ y_k \} \) is a sequence of distinct points in \( Y \) then for each \( n \geq 1 \) there is an \( f_n \in C^*(Y) \) such that

(a) \( f_n(Y) \subseteq [0, 1] \),
(b) \( f_n(y_p) = 0 \), \( 1 \leq p < n \),
(c) \( f_n(y_n) = 1 \),
(d) \( f_n(y_p) \) is rational if \( p > n \).

Proof. By hypothesis there is a continuous \( g_0: Y \rightarrow [0, 1] \) such that \( g_0(y_p) = 0 \), \( 1 \leq p < n \), and \( g_0(y_n) = 1 \). We define a sequence \( \{ g_k \} \) as follows. If \( g_0, g_1, \ldots, g_{k-1} \) have been defined and \( g_{k-1}(y_{n+k}) = c \) then if \( c \in \{ 0, 1/2^{k-1}, 2/2^{k-1}, \ldots, (2^{k-1} - 1)/2^{k-1}, 1 \} \) define \( g_k = g_{k-1} \). Otherwise let \( g_k = h_k \circ g_{k-1} \) where, if \( c \in (p/2^{k-1}, p+1/2^{k-1}) \), then \( h_k \) is the homeomorphism of \( [0, 1] \) onto itself which is the identity mapping on \( [0, p/2^{k-1}] \cup \) \( [p+1/2^{k-1}, 1] \), sends \( [p/2^{k-1}, c] \) onto \( [p/2^{k-1}, 2p+1/2^k] \), and sends \( [c, p+1/2^{k-1}] \) onto \( [2p+1/2^k, p+1/2^{k-1}] \). \( \{ g_k \} \) is a uniformly convergent sequence since \( |g_k(x) - g_{k+p}(x)| \leq 1/2^k + 1/2^{k+1} + \ldots + 1/2^{k+p} < 1/2^{k-1} \). Thus, \( f_n = \lim g_k \) is continuous and clearly has the desired properties.

Theorem 2. If \( |\beta X \setminus X| \leq \kappa_0 \) then there is an \( f \in C^*(X) \) such that \( K(f) = \{ \beta X \} \).

Proof. Let \( \beta X \setminus X = \{ x_1, x_2, \ldots \} \) and let \( \{ f_n \} \) be the sequence guaranteed by the lemma. Define \( g: \beta X \rightarrow \mathbb{R}^1 \) by \( g(x) = \sum_n (\pi/4)^n f_n(x) \). Then \( g \in C^*(\beta X) \).
and if $f = g|_{\beta X}$ it follows that $f^{\beta} = g$. If $g(x_p) = g(x_q)$ then $\sum_p (\pi/4)^n f_n(x_p) = \sum_q (\pi/4)^n f_n(x_q)$, contradicting the transcendence of $\pi$ ($f_n(x_p), f_n(x_q)$ are all rational). Thus, $f^{\beta}$ is 1-1 on $\beta X \setminus X$ so that, by Theorem 1, $K(f) = \{\beta X\}$.

If $\aleph_0 < |\beta X \setminus X| \leq c$, we cannot obtain such a definitive result as can be seen by the following two examples.

**Example 1.** Let $X = W(\omega_1 + 1) \times W(\omega_1)$. Then $\beta X \setminus X$ is homeomorphic to $W(\omega_1 + 1)$ [1, pp. 137–138] so that no $f^{\beta}$ can be 1-1 $\beta X \setminus X$. Thus $K(f) \neq \{\beta X\}$ for any $f \in C^*(X)$. In this case $|\beta X \setminus X| = \aleph_1 \leq c$.

**Example 2.** Let $X = [0, 1] \times W(\omega_\alpha)$ where $\alpha$ is a nonlimit ordinal for which $\aleph_\alpha > c$. Then $\beta X = [0, 1] \times W(\omega_\alpha + 1)$ so that $\beta X \setminus X$ is homeomorphic to $[0, 1]$. Let $f: X \to R$ be the projection onto the first coordinate. Clearly $f^{\beta}|_{\beta X \setminus X} = \text{id}_{[0, 1]}$ so that $K(f) = \{\beta X\}$.

**Reference**


DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607