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PRO-NILPOTENT REPRESENTATION OF HOMOLOGY TYPES

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ABSTRACT. The completion tower R_nX defined by Bousfield and Kan is shown to preserve the homology with R-coefficients. This property of preserving R-homology characterizes the tower completely [4].

0. **Introduction.** For a given space X and a solid ring R (i.e., a commutative ring with 1 for which the multiplication map $R\otimes R\to R$ is an isomorphism), Bousfield and Kan define $R_\infty X$—the R-completion of X. The functor $R_\infty X=\text{proj lim } R_nX$ [3] has very nice properties and generalizes all the other partial and nonfunctorial construction of completion and localizations.

One of the most important unsolved problems with regard to R_∞ is the relationship between $H_*(X, R)$ and $H_*(R_\infty X, R)$. It is known [2] that if X is an infinite wedge of circles then the natural map $H_*(X, Z)\to H_*(Z_\infty X, Z)$ is *not* an isomorphism. But nothing is known if X is, e.g., a wedge of two circles.

The purpose of this note is to show that one always gets a "pro-homology isomorphism" if one substitutes for $R_\infty X$ the whole tower $(R_nX)_n$. In fact, we will prove that for any fixed $k\geqq 0$ the map $H_k(X, R)\to (H_k(R_nX, R))_n$, where the target is a tower of abelian groups—considered as a pro-group, is an isomorphism of pro-groups. In particular,

$$H_k(X, R) \xrightarrow{\simeq} \operatorname*{proj\,lim}_n H_k(R_nX, R); \qquad \operatorname*{proj\,lim}_n{}^1 H_k(R_nX, R) = 0.$$

This is equivalent to

$$\operatorname*{inj\,lim}_n H^k(R_nX, M) = H^k(X, M) \quad \text{for any } R \text{ module } M.$$

1. **Pro-groups, pro-homotopy type.** We will consider the category whose objects are towers:

$$(K_n)_n = \cdots \to K_i \to K_{i-1} \to \cdots \to K_0$$

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of simplicial sets with one vertex or of groups. The morphisms are given by

$$\operatorname{Hom}((K_n)_n, (L_m)_m) = \operatorname*{proj\,lim}_m \operatorname*{inj\,lim}_n \operatorname{Hom}(K_n, L_m)$$

and can be thought of as a commutative ladder $(K_n)_n \to (L_m)_m$.

A short introduction to the category of pro-objects is given in the Appendix of [**1**], in which a theory of pro-homotopy type is developed. Another approach to the homotopy type of diagram is given in [**4**]. A more extensive treatment of pro-objects over an abelian category is given in [**6**], [**4**].

The *pro-nilpotent completion* of a tower $(X_n)_n$ of "connected spaces", i.e., of simplicial sets with one vertex, is given by the tower R_nX_n where the map $R_nX_n \to R_{n-1}X_{n-1}$ is the composition $R_nX_n \to R_nX_{n-1} \to R_{n-1}X_{n-1}$. Thus for every such pro-simplicial set X_n one has a natural map of towers:

$$(X_n) \to (R_nX_n).$$

2. THEOREM. *Let $(X_n)_n$ be a tower of "connected spaces"; then for every $k \geqq 0$ the map $(H_k(X_n, R))_n \to (H_k(R_nX_n, R))_n$ is an isomorphism in the category of pro-groups.*

2.1. LEMMA. *The map $(X_n)_n \to (R_nX_n)_n$ induces an isomorphism of R-homology if and only if the map $(R_nX_n)_n \to (R_nR_nX_n)_n$ is a homotopy equivalence of pro-simplicial sets.*

PROOF. The argument is the same as in [**4**]: One looks at the square of abelian pro-groups

$$\begin{array}{ccc} RX_n & \longrightarrow & RR_nX_n \\ \psi\big\uparrow\big\downarrow\varphi & & \psi\big\uparrow\big\downarrow\varphi \\ RR_nX_n & \longrightarrow & RR_nR_nX_n \end{array}$$

which exists and commutes by virtue of the triple structure of the functor R_s.

Since the bottom map is an equivalence and $\psi\varphi = \mathrm{id}$, it follows that the top map is an equivalence also, i.e., $H_k(X_n, R) \simeq \pi_k RX_n \to H_k(R_nX_n, R)$ is an isomorphism of pro-groups.

2.2. PROOF OF THE THEOREM. According to Lemma 2.1 it suffices to prove that the map $(R_nX_n)_n \to (R_nR_nX_n)_n$ is a homotopy equivalence of towers. Since the tower of the target is cofinal [**1**] in the pro-simplicial set $(R_sR_nX_n)_{s,n}$, it is enough to show that for any *fixed n* the map

$$(*) \qquad R_nX_n \xrightarrow{\sim} (R_sR_nX_n)_s \qquad (n\text{-fixed})$$

is a homotopy equivalence of towers, when we regard R_nX_n as a constant tower.

We prove (*) by showing that for every *fixed* n, $k \geqq 0$, the map $\pi_k R_n X \to (\pi_k R_s R_n X_n)_s$ is an isomorphism of pro-groups.

One first notes that, as A. K. Bousfield observed, it follows from the convergence of the homotopy spectral sequence [4] that for any $k, s \geqq 0$ there exists $N = N(s, Y, k)$ such that

$$\mathrm{Image}(\pi_k R_\infty Y \to \pi_k R_s Y) = \mathrm{Image}(\pi_k R_n Y \to \pi_k R_s Y).$$

If one puts $Y = R_n X$ which is R-nilpotent, one gets $Y \xrightarrow{\simeq} R_\infty Y$. Thus it follows from the diagram

$$\begin{tikzcd}
& R_\infty R_n X \arrow[dl, shift left] \arrow[dd] & \\
R_n X_n \arrow[ur, shift left] \arrow[dr, shift left] & & (s \geqq n) \\
& R_s R_n X_n \arrow[ul, shift left] &
\end{tikzcd}$$

that for $s \geqq n$,

$$\mathrm{Im}(\pi_* R_\infty R_n X_n \to \pi_* R_s R_n X_n) \simeq \pi_* R_n X_n.$$

Thus, for some N,

$$\mathrm{Im}(\pi_* R_n R_n X_n \to \pi_* R_s R_n X) \simeq \pi_* R_n X.$$

Now for $k > 1$, it follows from the triple structure of R_n that for each $s \geqq n$ one has a natural decomposition

$$\pi_k R_s R_n X_n \simeq \pi_k R_n X_n \oplus J_s.$$

Thus one gets from the above that for any $s \geqq n$ there exists $N \geqq s$ such that the map $J_N \to J_s$ is the trivial map, which means that the pro-object $(\pi_k R_s R_n X)_s \simeq (\pi_k R_n X \oplus J_s)_s$ is isomorphic to the constant pro-object $\pi_k R_n X$ (see §5 of [6]).

As for the case $k = 1$, one uses the same argument for the quotient $\Gamma_r / \Gamma_{r+1} \pi_1$ of the lower central series and the fact that π_1 is nilpotent, i.e., that series is finite, to arrive at the same conclusion. This completes the proof.

3. **Corollaries.** We will now restrict ourself to the case when $(X_n)_n$ is a constant tower, i.e., $X_n = X$.

3.1. COROLLARY. *For every connected space X and $k \geqq 0$, $H_k X \to (H_k R_n X)_n$ is an isomorphism of pro-groups.*

Thus in particular one has the isomorphisms of groups:

$$H_k X \xrightarrow{\simeq} \operatorname*{proj\,lim}_n H_k R_n X, \qquad 0 \xrightarrow{\simeq} \operatorname*{proj\,lim^1}_n H_k R_n X.$$

3.2. *Pro-nilpotent completion à la Artin-Mazur.* One can consider the functor which assigns to every map $X \to N$ where N is an R-nilpotent space the space N, and to every commutative triangle

$$\begin{tikzcd} & N \arrow[dd] \\ X \arrow[ur] \arrow[dr] & \\ & N' \end{tikzcd}$$

with N, N' nilpotent spaces the map $N \to N'$. Thus one gets a pro-object $N_R X$ in the sense of [**1**], because the category of maps $X \to N$ with morphism the above triangles is a filtering category [**1**]: The product of two R-nilpotent spaces are nilpotent and so is the connected equalizer of two maps $N \rightrightarrows N'$ [**5**].

3.3. PROPOSITION. *For every connected space X, the tower $R_n X$ is cofinal in the pro-object $N_R X$.*

PROOF. Clearly $X \to N_R X$ is R-cohomology isomorphism since $k(R, n)$ are R-nilpotent. Since $X \to R_n X$ is R-homology isomorphism it is R-cohomology isomorphism. Since all the target spaces are R-nilpotent one gets a map $R_n X \to N_R X$ of pro-objects, which induces isomorphism on homotopy groups by the generalized Whitehead theorem [**5**] applied to towers of nilpotent spaces.

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