THE COMMUTATOR SUBGROUP MADE ABELIAN

JOEL M. COHEN

Abstract. A theorem on covering spaces is proved which yields the following information about a group \( \pi \), its commutator subgroup \( \pi' \) and their abelianizations: If \( \pi^{ab} \cong \mathbb{Z}_p^n \), a cyclic group of order a power of the prime \( p \), then \( \pi^{ab} = p\pi^{ab} \). Hence if \( \pi \) is also finitely generated, then \( \pi^{ab} \) is finite of order prime to \( p \).

The purpose of this note is to prove the following theorem and some related results:

Theorem 1. Let \( X \) be a connected CW complex with \( H_1(X) \cong \mathbb{Z}_p^n \). Let \( X' \) be the normal \( p^n \)-fold covering space of \( X \) with transformation group \( \mathbb{Z}_p^n \). Then \( H_1(X') \) is \( p \)-divisible. In particular, if \( \pi_1X \) is finitely generated, then \( \pi_1X' \) is also so \( H_1(X') \) is finite of order prime to \( p \).

Notation. If \( \pi \) is a group, \( \pi' = [\pi, \pi] \) is the commutator subgroup and \( \pi^{ab} = \pi / \pi' \) is its abelianization.

An immediate corollary is

Theorem 2. If \( \pi^{ab} \cong \mathbb{Z}_p^n \), a cyclic group of order a power of the prime \( p \), then \( \pi^{ab} \) is \( p \)-divisible; i.e. \( \pi^{ab} = p\pi^{ab} \).

Theorem 2 follows from Theorem 1 by observing that the Eilenberg-Mac Lane space \( K(\pi', 1) \) is the \( p^n \)-fold covering space of \( K(\pi, 1) \) and \( H_1(K(\pi', 1)) = \pi^{ab} \), \( H_1(K(\pi', 1)) = \pi^{ab} \).

The proof of Theorem 1 is based on the homology Serre Spectral Sequence of the fibration \( X' \to X \to K(\mathbb{Z}_p^n, 1) \): \( E^2_* = H_*(\mathbb{Z}_p^n; H_*(X')) \) (local coefficients based on the action of \( \mathbb{Z}_p^n \) on \( X' \)) converging to \( H_*(X) \) (simple \( \mathbb{Z} \)-coefficients).

Because it is a first quadrant spectral sequence there is an exact sequence

\[
E^2_{2,0} \to E^2_{0,1} \to H_1(X) \to E^2_{1,0} \to 0.
\]
But $E_{s,0}^2 = H_s(Z_{p^n}; H_0(X')) = H_s(Z_{p^n})$ which is $Z_{p^n}$ for $s = 1$ and 0 for $s = 2$. Since $H_1(X) \to H_1(Z_{p^n})$ is an isomorphism, we conclude that

$$H_0(Z_{p^n}; H_1(X')) = E_{0,1}^2 = 0.$$  

The theorem is proved once we show:

**Proposition.** If $G$ is a finite $p$-group and $H_0(G; M) = 0$ for some $G$-module $M$, then $M$ is $p$-divisible.

**Proof.** $H_0(G; M) = M/[M]$ where $I = \ker \epsilon : Z[G] \to Z$ is the augmentation. So $H_0(G; M) = 0$ means $M = IM$. The proposition will be proved if we can show that for some integer $N$, $I^N \subseteq pI$, whence $M = I^N M \subseteq pIM = pM$. This is equivalent to showing that $J = I \otimes Z_p = \ker \epsilon \otimes Z_p$ is nilpotent. This is well known [1, p. 703] but for completeness we prove it here for the case $G = Z_n$: Let $t$ be the generator of $Z_{p^n}$ (written multiplicatively). $t^{p^n} = 1$. Then $J = (t-1)Z_{p^n}G$. $J^{p^n} = (t-1)^{p^n}Z_{p^n}G$. But modulo $p$, $(t-1)^{p^n} \equiv (t^{p^n} - 1) \equiv 0$ so $J^{p^n} = 0$.

**Reference**


**Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104**