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COVERINGS OF INFINITE-DIMENSIONAL SPHERES

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ABSTRACT. Let F be a normed linear space such that the countable infinite product of F is homeomorphic to a normed linear space. (This is true for all Hilbert spaces, for example.) Let $S(F)$ denote the unit sphere in F. We prove the following

THEOREM 1. *There is a countable cover of $S(F)$ of open sets each of which contains no pair of antipodal points.*

THEOREM 2. *There is a countable collection of closed sets in $S(F)$ the union of which contains exactly one member of each pair of antipodal points.*

THEOREM 3. *Let F be a Hilbert space. Then there is a countable collection of sets which cover $S(F)$ and whose diameters are less than* 2.

Let F be a normal linear space, and let $S(F)$ denote the unit sphere in F. We will prove several theorems in the case where F^ω (the countable infinite product of F) is homeomorphic to a normed linear space. Let us call this condition Property A. We note that Property A is satisfied by all finite-dimensional spaces (since s, the countable infinite product of lines, is homeomorphic to l_2, separable infinite-dimensional Hilbert space), and all Hilbert spaces (see **[1]** or **[4]**), and hence all spaces homeomorphic to a Hilbert space, such as reflexive Banach spaces **[2]**. The following theorems are proved for normed linear spaces with Property A.

THEOREM 1. *There is a countable cover of open sets of $S(F)$ each of which contains no pair of antipodal points.*

THEOREM 2. *There is a countable collection of closed sets in $S(F)$ the union of which contains exactly one member of each pair of antipodal points.*

THEOREM 3. *Let F be a Hilbert space. Then there is a countable collection of sets which cover $S(F)$ and whose diameters are less than* 2.

First, we note that Theorem 2 is proved by Klee for normed linear spaces of vector space dimension $\leqq c$ (the cardinality of the continuum) (Corollary 2.7 of [7]). We generalize this to spaces of all dimensions with the restriction that F have Property A, and give a completely different proof. Theorem 3 is related to the following question.

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QUESTION. *Let F be Hilbert space of Schauder dimension c. Can $S(F)$ be covered by a countable collection of sets whose diameters are uniformly less than* 2?

This question has important applications in graph theory discovered by Erdös. However, the methods of this paper do not suggest a solution to the question.

Restriction to the case $F \cong F^\omega$. We will now show that we need only prove the theorems for the case $F \cong F^\omega$. Since Theorems 2 and 3 are proved as corollaries to Theorem 1, we will only look at Theorem 1. Let G be the normed linear space homeomorphic to F^ω. Look at the space $F \times G$ with norm $\|(x, y)\| = \|x\|_F + \|y\|_G$. Now $S(F)$ is naturally imbedded in $S(F \times G)$ as $S(F) \times \{0\}$, with antipodal points preserved, hence we need only prove Theorem 1 for $F \times G$. But $(F \times G)^\omega \cong (F \times F^\omega)^\omega \cong F^\omega \cong F \times F^\omega \cong F \times G$.

Therefore, throughout the rest of the paper, we will assume that $F \cong F^\omega$. (It is unknown whether this condition holds for all infinite-dimensional Banach spaces.) We note that $F \cong l_2 \times F$. This follows easily from the facts that $F \cong F^\omega$ and F contains R as a topological factor. Also, $S(F) \cong F$ (see [3]).

Throughout the paper, $P(F)$ will denote $S(F)$ with antipodal points identified (projective F space). We remark that the fundamental group of $P(F)$ is $Z/2$ and all other homotopy groups are trivial. The projection map from $S(F)$ to $P(F)$ will be denoted by π.

Several lemmas.

LEMMA 1. *If U is an open contractible set in $S(F)$ containing no pair of antipodal points, then $\pi(U)$ is contractible.*

PROOF. $\pi: U \to \pi(U)$ is a homeomorphism.

LEMMA 2. *There is a countable cover of $P(l_2)$ by contractible open sets.*

PROOF. Let $S(l_2) = \{(x_1, x_2, \cdots) | x_i \in R, \sum x_i^2 = 1\}$. Let

$$U_i = \{(x_1, x_2, \cdots) \in S(l_2) \mid x_i > 0\}$$

and

$$U_i' = \{(x_1, x_2, \cdots) \in S(l_2) \mid x_i < 0\}.$$

Now let $V_i = \pi(U_i)$. $\pi^{-1}(V_i) = U_i \cup U_i'$ and the U_i's and U_i''s cover $S(l_2)$. Hence the V_i cover $P(l_2)$ and by Lemma 1 are contractible.

LEMMA 3. $P(F) \cong P(l_2) \times F$.

PROOF. $P(F)$ has the same homotopy groups as $P(l_2) \times F$. Both are F-manifolds, and hence are ANR's. Hence they both have the homotopy type of a CW complex (p. 135 of [8]). The CW complexes are Eilenberg-Mac Lane spaces (specifically $K(Z/2, 1)s$), hence by p. 82 of [9] have the same homotopy type. But by [6], all homotopy-equivalent F-manifolds are homeomorphic.

LEMMA 4. *There is a countable cover of $P(F)$ by contractible open sets.*

PROOF. Let V_i be the sets of Lemma 2 and look at $V_i'=V_i\times F$. If $h:P(F)\to P(l_2)\times F$ is the homeomorphism of Lemma 3, then $h^{-1}(V_i')$ are the desired sets.

LEMMA 5. *Let U be an open contractible set in $P(F)$. Then $\pi^{-1}(U)$ is the disjoint union of two open sets in $S(F)$ which are antipodals of each other.*

PROOF. Obviously, $\pi^{-1}(U)$ has either one or two components. If it has only one, then it is path-connected since we are in a manifold. Connect a pair of antipodal points with a path in $\pi^{-1}(U)$, say L. Then $\pi(L)$ is the image of a generator of the fundamental group of $P(F)$ which is $Z/2$. Hence U is not contractible.

LEMMA 6. *Let F be a Hilbert space. There is a function $f:(0, 2)\to(0, 2)$ such that if $\varepsilon_i\to 0$ then $f(\varepsilon_i)\to 0$ satisfying: If $x, y\in S(F)$ and $d(x, -y)>f(\varepsilon)$ then $d(x, y)<2-\varepsilon$.*

PROOF. Routine using trigonometry and the "roundedness" of Hilbert space spheres.

Proof of the theorems.

PROOF OF THEOREM 1. Let U_i be the cover of $P(F)$ by contractible open sets guaranteed by Lemma 4. Using Lemma 5, let V_i, V_i' be the components of $\pi^{-1}(U_i)$ and we are done.

PROOF OF THEOREM 2. Let V_i, V_i' be as in the proof of Theorem 1. Write V_i as the countable union of closed sets $C_{i,n}$. Let $C'_{i,n}=C_{i,n}-\bigcup_{j=1}^{i-1} V_j'$. Then the $C'_{i,n}$ are the desired sets.

PROOF OF THEOREM 3. Let V_i be the open sets in Theorem 1 for F a Hilbert space. Let $V_{i,n}=\{x\in F\,|\,d(x, F-V_i)>f(1/n)\}$ where f is the function from Lemma 6. Then the diameter of $V_{i,n}$ is $2-1/n$ or less and $\bigcup_n V_{i,n}=V_i$.

We complete the paper with an alternative proof of the theorems for Hilbert spaces which avoids the high-powered results used in the proof of Lemma 3. We will prove Theorem 1 and the others follow as above.

Let $H(A)=\{f:A\to R\,|\,\sum_{a\in A}(f(a))^2<\infty\}$ where f need not be continuous. (A is just an index set on the coordinates.) Since A is infinite, let B be a countably infinite subset of A. Let

$$U_b = \{f: A \to R \mid f(b) > 0\} \cap S(H(A)),$$
$$U'_b = \{f: A \to R \mid f(b) < 0\} \cap S(H(A)) \quad \text{for } b \in B.$$

Let $K=\{f: A\to R | f|B=0\} \cap S(H(A))$. The sets U_b, U'_b are a countable collection of open sets whose union is $S(H(A))-K$.

Now we can regard $S(H(A))$ as $S(H(B)) * S(H(A-B))$ where $*$ denotes join. Since $S(H(B)) \simeq l_2$ then $(S(H(A)), K)$ looks locally like $(H(A-B)\times l_2, H(A-B)\times\{0\})$ and K has local infinite deficiency in $S(H(A))$. Now we look at $P(H(A))$. It is an $H(A)$-manifold and $\pi(K)$ has local infinite deficiency. (K has local infinite deficiency in X if every point of K has an open neighborhood U and a homeomorphism $h: U\to l_2\times Z$, for some Z, such that $h(U\cap K)\subset l_2\times \text{pt.}$) By the main theorem of [**5**], $\pi(K)$ is negligible, i.e. there is a homeomorphism $f: P(H(A))-\pi(K)\to P(H(A))$. Looking at the sets $\pi^{-1}f\pi(U_b)$ and applying Lemmas 1 and 5, we are done.

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