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SCALAR CURVATURE, INEQUALITY AND SUBMANIFOLD

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ABSTRACT. Using an inequality relation between scalar curvature and length of second fundamental form, we may conclude that a submanifold must have nonnegative (or positive) sectional curvatures. An application to compact submanifolds in obtained.

1. **Statement of results.**[1] Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold N of constant sectional curvature c, and let h and H be the second fundamental form and the mean curvature vector field respectively. Let h_{ij}^{α}, $i, j=1, \cdots, n$, $\alpha=n+1, \cdots, n+p$, be the coefficients of the second fundamental form h with respect to a local field of orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$. Then the square of length of second fundamental form, S, and the scalar curvature, R, of M are given respectively by

$$S = \sum_{\alpha=n+1}^{n+p} \sum_{i,j=1}^{n} (h_{ij}^{\alpha})^2, \tag{1}$$

$$R = n^2 H \cdot H - S + n(n-1)c, \tag{2}$$

where dot "·" denotes the scalar product of vectors. A normal vector field η is said to be parallel if $D\eta=0$ identically, where D denotes the connection of the normal bundle. The purpose of this paper is to show the following

THEOREM 1. *Let M be an n-dimensional submanifold of a Riemannian manifold N of constant curvature c. If the scalar curvature R satisfies*

$$R \geqq (n-2)S + (n-2)(n-1)c \qquad (\textit{resp. } R > (n-2)S - (n-2)(n-1)c) \tag{3}$$

at a point $p \in M$, then the sectional curvatures of M are nonnegative (resp. positive) at p.

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[1] For notations and formulas we refer to **[2]**.

THEOREM 2. *Let M be an n-dimensional compact submanifold of euclidean $(n+p)$-space E^{n+p}. Then the mean curvature vector H is parallel and we have $R>(n-2)S$ if and only if M is a hypersphere of a linear $(n+1)$-subspace of E^{n+p} when $n\geqq 3$, and M is a minimal surface of a hypersphere of E^{n+p} with positive Gaussian curvature when $n=2$.*

REMARK 1. If the connection of the normal bundle is flat, $n>2$, or if the submanifold is a hypersurface, Theorem 2 was proved by one of the present authors ([**3**], [**4**]).

2. **Proof of Theorem 1.** First we state the following lemma which is a slight generalization of a lemma given in [**4**]. The method of the proofs are quite the same.

LEMMA. *Let $a_1, \cdots, a_n$, b be $n+1$ $(n\geqq 2)$ real numbers satisfying the following inequality*:

$$(4)\qquad \left(\sum_{i=1}^{n} a_i\right)^2 \geqq (n-1)\sum_{i=1}^{n} a_i^2 + b \quad (resp. >);$$

then, for any distinct i, j; $1\leqq i<j\leqq n$, we have

$$(5)\qquad 2a_ia_j \geqq b/(n-1) \quad (resp. >).$$

This lemma is proved in the following way: (4) can be rewritten as

$$(n-2)a_n^2 - 2\left(\sum_{i=1}^{n-1} a_i\right)a_n + \left[(n-2)\sum_{i=1}^{n-1} a_i^2 - 2\sum_{i<j<n} a_ia_j + b\right] \leqq 0,$$

(resp. $<$). Denote the left-hand side by $-r$. Since a_n is real,

$$\begin{aligned}\left(\sum_{i=1}^{n-1} a_i\right)^2 &\geqq (n-2)\left[(n-2)\sum_{i=1}^{n-1} a_i^2 - 2\sum_{i<j<n} a_ia_j + b + r)\right]\\ &\geqq (n-2)\left[(n-1)\sum_{i=1}^{n-1} a_i^2 - \left(\sum_{i=1}^{n-1} a_i\right)^2 + b\right].\end{aligned}$$

Hence we obtain

$$\left(\sum_{i=1}^{n-1} a_i\right)^2 \geqq (n-2)\sum_{i=1}^{n-1} a_i^2 + \left(\frac{n-2}{n-1}\right)b \quad (\text{resp.} >).$$

Continuing the same process $(n-2)$ times, we obtain (5).

Substituting (2) into (3), we obtain

$$(6)\qquad n^2H\cdot H \geqq (n-1)S - 2(n-1)c \quad (\text{resp.} >) \quad \text{at } p.$$

For simplicity we may choose a local field of orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+p}$ around p such that e_{n+1} is parallel to the mean curvature vector H and $e_1, \cdots, e_n$ are in the principal directions of e_{n+1} at

$p \in M$. (If $H=0$ at p, we may choose an arbitrary e_{n+1}.) Then we have

$$(7)\qquad (h_{ij}^{n+1}) = \begin{pmatrix} h_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & \cdot & \\ & & & h_n \end{pmatrix}, \qquad n^2 H \cdot H = \left(\sum_{i=1}^{n} h_i\right)^2 \text{ at } p.$$

Thus we obtain from (6):

$$(8)\qquad \left(\sum_{i=1}^{n} h_i\right)^2 \geqq (n-1)\sum_{i=1}^{n} h_i^2 + (n-1)\sum_{\alpha=n+2}^{n+p}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 - 2(n-1)c$$

(resp. $>$).

Applying the lemma to (8), we get

$$(9)\qquad \begin{aligned} 2h_i h_j &\geqq \sum_{\alpha=n+2}^{n+p}\sum_{k,m=1}^{n}(h_{km}^{\alpha})^2 - 2c \\ &\geqq \sum_{\alpha=n+2}^{n+p}[(h_{ii}^{\alpha})^2 + (h_{jj}^{\alpha})^2 + 2(h_{ij}^{\alpha})^2] - 2c \\ &\geqq 2\sum_{\alpha=n+2}^{n+p}[h_{ii}^{\alpha}h_{jj}^{\alpha} + (h_{ij}^{\alpha})^2] - 2c, \end{aligned}$$

for any $1 \leqq i < j \leqq n$ at p. Thus the sectional curvature at p,

$$K_{ij} = \sum_{\alpha=n+1}^{n+p}[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2] + c,$$

for the plane section spanned by e_i and e_j is nonnegative (resp. positive). This proves the theorem.

3. **Proof of Theorem 2.** Let M be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold $\bar{N}$ of constant sectional curvature c and η be a parallel unit normal vector over M. If we choose the local fields of orthonormal frame in such a way that $e_{n+1}=\eta$ and $e_1, \cdots, e_n$ are in the principal directions of e_{n+1}, then we have

$$H_{n+1} = \begin{pmatrix} h_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & h_n \end{pmatrix}.$$

We assume that Tr H_{n+1} is constant. Then a recent paper of Smyth [5] gives the following formula:

$$(10)\qquad \sum_{i,j=1}^{n} h_{ij}^{n+1}\,\Delta h_{ij}^{n+1} = \sum_{i<j}\left[K_{ij} + \sum_{\beta}(h_{ij}^{\beta})^2\right](h_i - h_j)^2,$$

where Δh_{ij}^{n+1} denotes the Laplacian of the second fundamental form h_{ij}^{n+1} in the direction of e_{n+1}. Now, suppose that M is an n-dimensional compact submanifold of E^{n+p} such that the mean curvature vector H is parallel and $R>(n-2)S$. Then, by Theorem 1, we see that the sectional curvatures of M are all positive, that is, $K_{ij}>0$ for $1\leqq i<j\leqq n$. Therefore, we see that $\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1}\geqq 0$. Hence we get

$$\frac{1}{2}\Delta(\operatorname{Tr} H_{n+1}^2)=\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1}+\sum_{i,j,k}(h_{ijk}^{n+1})^2\geqq 0. \tag{11}$$

By Hopf's lemma we see that $h_{ijk}^{n+1}=0$ and $\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1}=0$. Hence, from (10) we have

$$h_1=\cdots=h_n\neq 0. \tag{12}$$

This shows that M is pseudo-umbilical in E^{n+p} and H is parallel. Hence, we see that M is contained in a hypersphere S^{n+p-1} of E^{n+p} as a minimal submanifold (see, for instance, [**1**]). Without loss of generality, we may assume that S^{n+p-1} is of radius 1. Then, by the assumption, $R>(n-2)S$, we see that the square of the length of second fundamental form of M in S^{n+p-1}, say $\bar{S}$, satisfies

$$\bar{S}<n/(n-1). \tag{13}$$

Therefore, by a result of Chern-do Carmo-Kobayashi [**2**], we find that if $n\geqq 3$, then M must be totally geodesic in S^{n+p-1}. Hence M is a hypersphere of a linear $(n+1)$-subspace of E^{n+p}. If $n=2$, then the condition $R>(n-2)S$ implies that the Gaussian curvature of M is positive. This proves a part of the theorem. The remaining part is obvious.

References

1. B.-Y. Chen, *On the mean curvature of submanifolds of euclidean space*, Bull. Amer. Math. Soc. **77** (1971), 741–743. MR **44** #2177.

2. S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. of Chicago, Chicago, Ill., 1968), Springer, New York, 1970, pp. 59–75. MR **42** #8424.

3. M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor* (to appear).

4. ———, *Submanifolds and a pinching problem on the second fundamental tensor* (to appear).

5. B. Smyth, *Submanifolds of constant mean curvature* (to appear).

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