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TORSION THEORY AND ASSOCIATED PRIMES[1]

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ABSTRACT. A torsion theory partitions the spectrum of the base ring into two sets. Over a Noetherian ring, every suitable partition of the spectrum gives rise to one and only one torsion theory. It is possible to know whether a module is torsion or torsion-free by looking at its associated primes. The example of the polynomial torsion theory is developed.

Notations and terminology. All rings are commutative with 1. We denote by $\mathrm{Ass}_A(M)$ the set of *weakly associated primes* of an A-module M.

A prime $\mathfrak{p}$ is in $\mathrm{Ass}_A(M)$ if $\mathfrak{p}$ is minimal among the primes containing the annihilator $\mathrm{ann}_A(x)$ of a nonzero element x of M.

We denote by $\mathbf{Ass}_A(M)$ the set of *strongly associated primes* of an A-module M.

A prime $\mathfrak{p}$ is in $\mathbf{Ass}_A(M)$ if it is the annihilator $\mathrm{ann}_A(x)$ of a nonzero element x of M.

We denote by $\mathrm{Supp}_A(M)$ the *support* of an A-module M.

A prime $\mathfrak{p}$ is in $\mathrm{Supp}_A(M)$ if the localization $M_\mathfrak{p}$ is not null.

We send the reader to Lambek [5] for the definition and immediate properties of a torsion theory. We denote by $T(M)$ the *torsion submodule* of a module M.

1. Partition of the spectrum.

Throughout this section A is a ring with a torsion theory.

PROPOSITION 1.1. *Let M be be an A-module.*

(1) *If $A/\mathfrak{p}$ is not a torsion module, for every $\mathfrak{p}$ in $\mathrm{Ass}_A(M)$, then M is a torsion-free module.*

(2) *If M is a torsion-free module, then $A/\mathfrak{p}$ is torsion-free for every $\mathfrak{p}$ in $\mathbf{Ass}_A(M)$.*

PROOF. (1) Suppose that M is not a torsion-free module. The torsion submodule $T(M)$ must contain a nonzero element x and $A/\mathrm{ann}_A(x) \simeq Ax$,

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which is included in $T(M)$, is a torsion module. So is its homomorphic image $A/\mathfrak{p}$ for any prime $\mathfrak{p}$ containing $\mathrm{ann}_A(x)$, i.e. for some $\mathfrak{p}$ in $\mathrm{Ass}_A(M)$.

(2) Suppose that M is torsion-free, then so is $A/\mathrm{ann}_A(x) \simeq Ax$ for any nonzero element x of M. Thus $A/\mathfrak{p}$ is torsion-free for every $\mathfrak{p}$ in $\mathbf{Ass}_A(M)$.

COROLLARY 1.2. *Let $\mathfrak{p}$ be a prime of A. Either $A/\mathfrak{p}$ is a torsion module or $A/\mathfrak{p}$ is a torsion-free module.*

PROOF. Apply Proposition 1.1 to the module $A/\mathfrak{p}$, and recall that $\mathbf{Ass}_A(A/\mathfrak{p}) = \mathrm{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$.

THEOREM-DEFINITION 1.3. *The torsion theory partitions* Spec A *into two subsets $\boldsymbol{T}$ and $\boldsymbol{F}$:*

$\boldsymbol{T}$ is the set of primes $\mathfrak{p}$ such that $A/\mathfrak{p}$ is torsion;

$\boldsymbol{F}$ is the set of primes $\mathfrak{p}$ such that $A/\mathfrak{p}$ is torsion-free;

$\boldsymbol{T}$ is closed under specialization and $\boldsymbol{F}$ is closed under generization.

PROOF. It remains to prove that $\boldsymbol{T}$ is closed under specialization (its complement $\boldsymbol{F}$ is then, of course, closed under generization). Let $\mathfrak{p}$ be a prime in $\boldsymbol{T}$ and let $\mathfrak{q}$ be another prime containing $\mathfrak{p}$. Since $A/\mathfrak{q}$ is an homomorphic image of $A/\mathfrak{p}$, we know that $A/\mathfrak{q}$ is also a torsion module, i.e. $\mathfrak{q} \in \boldsymbol{T}$.

Proposition 1.1 can now be summarized in the diagram:

$$\mathrm{Ass}_A(M) \subset \boldsymbol{F} \Rightarrow M \text{ is torsion-free} \Rightarrow \mathbf{Ass}_A(M) \subset \boldsymbol{F}.$$

PROPOSITION 1.4. *Let M be a torsion module, then* $\mathrm{Supp}_A(M)$ *is included in $\boldsymbol{T}$.*

PROOF. Suppose there is a prime $\mathfrak{p}$ of $\mathrm{Supp}_A(M)$ in $\boldsymbol{F}$. The canonical map $M \to M_\mathfrak{p}$ is not trivial. On the other hand $\mathrm{Ass}_A(M_\mathfrak{p})$ contains only primes which are included in $\mathfrak{p}$ [**8**]. Since $\boldsymbol{F}$ is closed under generization, $\mathrm{Ass}_A(M_\mathfrak{p}) \subset \boldsymbol{F}$, so $M_\mathfrak{p}$ is torsion-free. But for any torsion module E and any torsion-free module F, the group of homomorphisms $\mathrm{Hom}_A(E, F)$ is null [**5**]. Hence M is not a torsion module.

Since $\boldsymbol{T}$ is closed under specialization, this proposition can be summarized in the diagram:

$$M \text{ is torsion} \Rightarrow \mathrm{Supp}_A(M) \subset \boldsymbol{T} \Leftrightarrow \mathrm{Ass}_A(M) \subset \boldsymbol{T}.$$

2. **Well-centered torsion theory.** Of special interest are the theories for which the associated primes of a module M carry all the information.

PROPOSITION 2.1. *Let A be a ring with a torsion theory and $(\boldsymbol{T}, \boldsymbol{F})$ the corresponding partition of* Spec A. *If every torsion-free module is such that* $\mathrm{Ass}_A(M) \subset \boldsymbol{F}$ *and if P is a module such that* $\mathrm{Ass}_A(P) \subset \boldsymbol{T}$ *then P is a torsion module.*

PROOF. If $\mathrm{Ass}_A(P) \subset \boldsymbol{T}$ then $\mathrm{Supp}_A(P) \subset \boldsymbol{T}$. Let M be a torsion-free module and f a homomorphism $f: P \to M$ and denote by $\mathrm{Ker}(f)$ the kernel of f, then $\mathrm{Ass}_A(P/\mathrm{Ker}(f)) \subset \mathrm{Supp}_A(P/\mathrm{Ker}(f)) \subset \mathrm{Supp}_A(P) \subset \boldsymbol{T}$ [**8**]. On the other hand $P/\mathrm{Ker}(f)$ is embedded in M, hence $\mathrm{Ass}_A(P/\mathrm{Ker}(f)) \subset \mathrm{Ass}_A(M) \subset \boldsymbol{F}$, this last inclusion since M is torsion-free. But then

$$\mathrm{Ass}_A(P/\mathrm{Ker}(f)) \subset \boldsymbol{T} \cap \boldsymbol{F} = \varnothing$$

and so $P/\mathrm{Ker}(f) = (0)$. For every torsion-free module M, the group of homomorphisms $\mathrm{Hom}_A(P, M)$ is trivial and thus P is a torsion module [**5**].

DEFINITION 2.2. We say a torsion theory is *well-centered* when the following conditions hold:

M is torsion-free if and only if $\mathrm{Ass}_A(M) \subset \boldsymbol{F}$;

M is torsion if and only if $\mathrm{Ass}_A(M) \subset \boldsymbol{T}$.

Now if we start with a partition of Spec A into two subsets $\boldsymbol{T}$ and $\boldsymbol{F}$, such that $\boldsymbol{T}$ is closed under specialization, we want to know if this partition corresponds to a torsion theory. For the purpose of the following proposition, we say that a torsion theory is *coarser* than another if its class of torsion modules is larger.

PROPOSITION 2.3. *Let* $(\boldsymbol{T}, \boldsymbol{F})$ *be a partition of* Spec A, *such that* $\boldsymbol{T}$ *is closed under specialization. There is at least one torsion theory corresponding to this partition. The coarsest of all these theories is such that its torsion modules are characterized by the inclusion* $\mathrm{Ass}_A(M) \subset \boldsymbol{T}$, *which is equivalent to* $\mathrm{Supp}_A(M) \subset \boldsymbol{T}$.

PROOF. If $\boldsymbol{T}$ is any subset of Spec A let us denote by $\mathscr{T}$ the class of modules such that $\mathrm{Supp}_A(M) \subset \boldsymbol{T}$. The elementary properties of the support show that, for any exact sequence $0 \to N \to M \to P \to 0$, M is in $\mathscr{T}$ if and only if N and P are in $\mathscr{T}$; $\mathscr{T}$ is also closed under direct sum. Thus $\mathscr{T}$ is a torsion class, in other words $\mathscr{T}$ gives rise to a torsion theory whose torsion modules are the modules belonging to $\mathscr{T}$. Now, if we suppose that $\boldsymbol{T}$ is closed under specialization, then $\mathrm{Supp}_A(A/\mathfrak{p}) \subset \boldsymbol{T}$ if and only if $\mathfrak{p}$ itself is in $\boldsymbol{T}$ and so $(\boldsymbol{T}, \boldsymbol{F})$ is the partition corresponding to this theory.

If any other torsion theory gives the same partition and if M is a torsion module for this theory then $\mathrm{Supp}_A(M) \subset \boldsymbol{T}$ (Proposition 1.4) and M is in $\mathscr{T}$.

DEFINITION 2.4. We say that a torsion theory is half-centered if the following holds:

$\mathrm{Ass}_A(M) \subset \boldsymbol{T}$ if and only if M is torsion.

We will give an example of a half-centered theory which is not well-centered. By giving an example of a theory which is not even half-centered we will show that there may be more than one theory for the same partition (§4).

3. **Noetherian rings.**

THEOREM 3.1. *Every torsion theory over a Noetherian ring is well-centered. Every partition of the spectrum into two sets, one of them closed under specialization, gives rise to one and only one torsion theory.*

PROOF. The first statement comes from Proposition 1.1, since for every module M

$$(+) \qquad \mathbf{Ass}_A(M) = \mathrm{Ass}_A(M).$$

Any suitable partition of the spectrum gives rise to at least one torsion theory. There is only one since all the theories corresponding to the same partition would have the same torsion modules.

The theorem is true for every ring such that the equality $(+)$ holds for every module M.

PROPOSITION 3.2 *Let A be a Noetherian ring with a torsion theory, let $(\boldsymbol{T}, \boldsymbol{F})$ be the corresponding partition of* Spec A, *let M be an A-module of finite type and $(0)=\bigcap_{i\in I} M_i$ a reduced primary decomposition of* (0) *in M, where I is a finite set of indices. Let J be the subset of indices j such that the associated prime of M_j is in $\boldsymbol{F}$; then $T(M)=\bigcap_{j\in J} M_j$ is the torsion submodule of M.*

PROOF. One may easily check that $\mathrm{Ass}_A(T(M)) \subset \boldsymbol{T}$ and

$$\mathrm{Ass}_A(M/T(M)) \subset \boldsymbol{F}.$$

Thus $T(M)$ is a torsion module and $M/T(M)$ is torsion-free.

4. **Polynomial torsion theory.** Let M be an A-module. We denote by $M[X]$ the module of formal polynomials with coefficients in M. Such a polynomial can be evaluated at any element of A. The polynomial X^2-X with coefficients in the $\boldsymbol{Z}$-module $\boldsymbol{Z}/2\boldsymbol{Z}$, for example, is null at every integer.

PROPOSITION-DEFINITION 4.1. *The A-modules M such that the polynomial* 0 *is the only polynomial in $M[X]$ which is null at all elements of A are the torsion-free modules for a torsion theory over A. We call polynomial torsion theory this torsion theory, and polynomial torsion-free these modules.*

It is easy to check [**3**] that this class of modules is closed under inclusion, direct product, essential extension and group extension, and this is enough to prove the proposition [**5**].

PROPOSITION 4.2. *The partition of the spectrum corresponding to the polynomial torsion theory is such that $\boldsymbol{T}$ is the set of primes $\mathfrak{p}$ such that $A/\mathfrak{p}$ is finite.*

PROOF. [**3**, §1, Example 2].

We now give two examples. In the first one the polynomial torsion theory is half-centered but not well-centered, in the second one this theory is not even half-centered.

EXAMPLE 4.3. Let B be the ring of locally constant functions from the reals $\boldsymbol{R}$ to an algebraic closure of a finite field k. Let A be the subring of B of all functions f such that $f(0) \in k$. The polynomial torsion theory over A is half-centered but not well-centered.

PROOF. It is easy to check that all the primes of A have infinite residue field but the prime $\mathfrak{p}_0$, consisting of the functions which are null at 0, which is such that $A/\mathfrak{p}_0 \simeq k$. $\boldsymbol{T}$ contains only the prime $\mathfrak{p}_0$. If M is a module such that $\mathrm{Ass}_A(M) \subset \boldsymbol{T}$, or equivalently $\mathrm{Supp}_A(M) \subset \boldsymbol{T}$, then M is a submodule of $M_{\mathfrak{p}_0}$, and $M_{\mathfrak{p}_0}$ is a k-vector space, that is to say a direct sum of copies of k. Since k is a torsion module, so is this direct sum and so is M. Thus the polynomial torsion theory is half-centered. But the A-module A itself is polynomial torsion-free. Indeed for any $f \neq 0$, $f \in A$, there is a prime $\mathfrak{p}$ distinct from $\mathfrak{p}_0$ containing $\mathrm{ann}_A(f)$. Since $A/\mathfrak{p}$ is torsion-free, Af cannot be a torsion module and thus $T(A)$ contains only 0. However, $\mathfrak{p}_0$ is a minimal prime of A, hence $\mathfrak{p}_0 \in \mathrm{Ass}_A(A)$. Thus the polynomial torsion theory is not well-centered.

EXAMPLE 4.4. Let A be a valuation ring of rank 1, K its field of quotients, v the valuation of K corresponding to A and G the value group of v. Suppose that G is a dense additive subgroup of the reals $\boldsymbol{R}$ and that the residue field of v is finite, then the polynomial torsion theory over A is not half-centered.

PROOF. We show that the module K/A is polynomial torsion-free. However $\mathrm{Ass}_A(K/A)$ consists of the maximal ideal of A, whose residue field is finite. To say that K/A is polynomial torsion-free amounts to saying that any polynomial P in $K[X]$, such that $P(A) \subset A$, has its coefficients in A. Let $P = k_0 + k_1X + \cdots + k_nX^n$ and let i be the smallest index between 1 and n such that $v(k_i) \leqq v(k_j)$, $\forall j$, $0 \leqq j \leqq n$. There is an element $g > 0$, $g \in G$, small enough such that $v(k_i) + ig < v(k_j) + jg$, $\forall j$, $0 \leqq j \leqq n$, and if $a \in A$ is such that $v(a) = g$, then since $P(A) \subset A$,

$$v(P(a)) = v(k_i) + ig \geqq 0.$$

Since g can be made arbitrarily small, then $v(k_i) \geqq 0$ and also $v(k_j) \geqq 0$, $\forall j$, $0 \leqq j \leqq n$. Thus $P \in A[X]$.

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