A NOTE ON PATHS THROUGH $O$

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Abstract. We show that a hyperarithmetic set can be truth table reduced to a $\Pi^1_1$-path through $O$ iff it is truth table reducible to some r.e. set.

It is known from results of Feferman and Spector [1] that while there are no $\Sigma^1_1$-paths through $O$, there do exist $\Pi^1_1$ such paths. Here by a path is meant a linearly ordered subset $P$ of $O$, closed under $<_O$ and having order type $\omega_1$.

In this note we prove

Theorem 1. If $P$ is a $\Pi^1_1$-path through $O$, $A$ is hyperarithmetic, and $A$ is truth table reducible to $P$, then the Turing degree of $A$ is at most $0'$. Thus in a certain sense such paths contain very little "information". It is not known if a hyperarithmetic set of Turing degree greater than $0'$ can be Turing reduced to such a path.

Proof. We shall actually prove a somewhat stronger fact. If $P$ is a $\Pi^1_1$-path through $O$ and $A$ is truth table reducible to $P$, then either $A$ is truth table reducible to a proper segment of $P$ (which is necessarily r.e.) or else all of $P$ is used in an essential way, and then one can go backwards and arithmetically decide $P$ from $A$.

So suppose the Turing degree of $A$ is not $\leq 0'$ and $A$ is truth table reducible to such a path $P$. Then there exists a $z$ such that, for all $n$, $C_A(n) = U(\mu y T^P(z, n, y))$ and moreover, for all $X \subseteq N$, $U(\mu y T^X(z, n, y))$ is a total function of $n$. (See [4, p. 143, Theorem XIX].)

Now by (1) there exists a $d \in O^*$ and an r.e. ordering $\leq$ such that $P \subseteq P_1 = \{y | y \leq d\}$, $\leq$ is a linear ordering on $P_1$ and $P$ is the maximal well ordered segment of $P_1$. (In fact, $\leq$ is that r.e. linear ordering which restricts on $O$ to $<_O$.)

Consider pairs $(a, b)$, $a, b \in P_1$, $a < b$, and consider a $z$-computation of $C_A(n)$ where: whenever the machine asks, "does $m \in P$?" the answer given is "yes" if $m \leq a$, "no" if $m \notin P_1$ or $b \leq m$, and no answer is given if neither condition holds. We will say that $(a, b)$ is adequate for $n$ if the correct value of $C_A(n)$ is computed in this way.

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Clearly, there is an $(a, b)$ adequate for a given $n$ and in fact for finitely many $n$. For consider the actual computations of $C_A(n_i)$, $i=1, \ldots, k$, from $P$ itself. Let $a=\max$ relative to $\leq$ of all yes answers given and $b=\min$ of all no answers, then $(a, b)$ is adequate for $n_1, \ldots, n_k$. In this case we shall have $a \in P, b \notin P$. However, it is conceivable that $(a, b)$ is adequate for the $n_i$'s and yet both (or neither) of $a, b$ are in $P$. Note that if $(a, b)$ is adequate for $n$, $a < a' < b' \leq b$ then $(a', b')$ is adequate for $n$.

We will say that $(a, b)$ is good, if for all $F \subseteq N$, $F$ finite, there are $a', b'$, $a \leq a' < b' \leq b$ such that $(a', b')$ is adequate for (every element of) $F$. The predicate $G(a, b)$ ($(a, b)$ is good) is arithmetical in $A$ and hence hyperarithmetic.

Now we claim that if $G(a, b)$, then $\exists X, \{y | y \leq a\} \subseteq X \subseteq \{y | y \leq b\}$ and $C_A(n) = U(\mu y T^X(z, n, y))$.

For note that if $a < c < b$ and $G(a, b)$ then either $G(a, c)$ or $G(c, b)$. Otherwise there are finite sets $F_1, F_2$ such that no pair $(a', b')$ contained in $(a, c)$ is adequate for $F_1$, and no pair $(a', b')$ contained in $(c, b)$ is adequate for $F_2$. But then no pair $(a', b')$ contained in $(a, b)$ is adequate for $F_1 \cup F_2$ contradicting the goodness of $(a, b)$. (Either $(a', b')$ is contained in $(a, c)$ or in $(c, b)$ or else $(a', c)$ is adequate for $F_1$.)

Now given $G(a, b)$, let $c_1, c_2, \ldots$ be an enumeration of $P$. Define $(a_i, b_i)$ by:

$(a_0, b_0) = (a, b)$,
$(a_{n+1}, b_{n+1}) = (a_n, b_n)$ unless $a_n < c_{n+1} < b_n$,
$(a_{n+1}, c_{n+1})$ if $a_n < c_{n+1} < b_n$ and $G(a_n, c_{n+1})$,
$(c_{n+1}, b_n)$ otherwise.

Then $G(a_n, b_n)$ holds for all $n$ and no $x$ can satisfy $a_n < x < b_n$ for all $n$. Let $X$ be the set $\{x | (\exists n)(x \leq a_n)\}$. Claim $U(\mu y T^X(z, n, y))$ always equals $C_A(n)$.

Otherwise (by choice of $z$) there is a $p$ such that $U(\mu y T^X(z, p, y))$ is defined and unequal to $C_A(p)$. Consider $a$, the maximum (in $\leq$) of all the “yes” answers during this computation from $X$. Similarly $b$ is the minimum of all “no” answers for elements of $P$. Then there is an $n$ such that $a \leq a_n < b_n \leq b$. But clearly $(a_n, b_n)$ cannot be good since no pair $(a', b')$ contained in $(a, b)$ can be adequate for $\{p\}$.

Thus $G(a, b)$ implies $b \notin P$. Otherwise an $X$ such as above would be a proper segment of $P$, and hence r.e. But clearly $b \in P_1 - P$ implies that for all $a \in P$, $G(a, b)$. Hence we get

$x \in P \leftrightarrow x \in P_1 \land \neg (\exists a)(G(a, x))$,

a contradiction since $P$ is not hyperarithmetic. Q.E.D.
It is fairly straightforward to show that the lower bound on the degree of $A$ is best possible.

**Theorem 2.** There is a $\Pi^1_1$-path $P$ through $O$ such that the set $A = \{n|(\exists y)T(n, n, y)\}$ is 1-1 reducible to $P$.

**Proof.** Define an ordering $R$ on $N$ by

$$R(x, z) \iff (\exists y)(T(x, x, y) \land (\forall u \leq y)(\neg T(z, z, u))).$$

This is a partial ordering, the elements in $A$ are a sequence of type $\omega$ and the elements not in $A$ are a set of incomparables above this sequence. Now apply the construction of [3, p. 45] to this ordering. We get a function $g$ from $N$ to $O$ such that $g[N-A]$ is a set of incomparables and if $x \in A$, $y \in N-A$ then $g(x) <_O g(y)$. Take any $x_0 \in N-A$ and pick $d \in O^*$ such that $g(x_0) <_O d$. The path $P$ corresponding to $d$ has the required property.

**Remark.** Note that $A$ is in fact 1-1 reduced to a proper segment $S$ of $P$. $P-S$ plays no role, since $g[A] \subseteq S$ and $g[N-A] \subseteq N-P$.

**References**


2. G. Kreisel, *Which number theoretic problems can be solved in recursive progressions on $\Pi^1_1$-paths through $O$?*, J. Symbolic Logic 37 (1972), 311–334.


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1 This result was also found, independently and simultaneously by C. Jockusch.