ON THE RADIUS OF $\beta$-CONVEXITY OF STARLIKE FUNCTIONS OF ORDER $\alpha$

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Abstract. A function $f(z) = z + a_2z^2 + \cdots$ is called $\beta$-convex if $f(z)f'(z)/z \neq 0$ in $D: |z| < 1$ and if

$$\Re\{(1 - \beta)zf'(z)f(z) + \beta(1 + zf''(z)f'(z))\} > 0$$

for some $\beta \geq 0$ and all $z$ in $D$. Recently M. O. Reade and P. T. Mocanu have announced a sharp result about the radius of $\beta$-convexity for starlike functions. The author generalizes this result to starlike functions of order $\alpha$.

1. Introduction. Let $f(z) = z + a_2z^2 + \cdots$ be analytic in the unit disc $D: |z| < 1$. We say that $f(z)$ is starlike of order $\alpha$, $0 \leq \alpha < 1$, if

$$\Re\{zf'(z)/f(z)\} > \alpha$$

for all $z$ in $D$. We denote such a class of functions by $S_\alpha^*$. We say that $f(z)$ is convex of order $\alpha$, $0 \leq \alpha < 1$, if

$$\Re\{1 + zf''(z)f'(z)\} > \alpha,$$

for all $z$ in $D$. We denote such a class of functions by $C_\alpha$. For $\alpha = 0$, $S_0^*$, $C_0$ are simply called starlike and convex, respectively.

We consider now a class of functions which is formed by a linear combination of the conditions stated in (1) and (2).

Definition. Let $f(z) = z + a_2z^2 + \cdots$ be analytic in $D$ with $f(z)f'(z)/z \neq 0$ in $D$. Let

$$L(\beta; f) = (1 - \beta)zf'(z)f(z) + \beta(1 + zf''(z)f'(z)).$$

If

$$\Re\{L(\beta; f)\} > 0$$

Received by the editors June 22, 1972 and, in revised form, August 7, 1972.


Key words and phrases. Univalent functions, convex and starlike functions of order $\alpha$, $\beta$-convex functions, radius of convexity, radius of $\beta$-convexity, extremal functions.

1 The author presented a short talk based on the material of this note to the Regional Conference on Conformal and Quasiconformal Mappings at Kent State University, May 1972.

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for some \( \beta, \beta \geq 0, z \in D \), then \( f(z) \) is called a \( \beta \)-convex function. We denote this class by \( C(\beta) \).

Mocanu [2] was the first to introduce the class of \( \beta \)-convex functions under the restrictions \( 0 \leq \beta \leq 1 \) and that \( f(z) \) must be univalent in \( D \). Recently, however, Mocanu and Reade [3] have shown that each function in \( C(\beta) \) is univalent (starlike) for \( \beta \geq 0 \). In particular each \( f(z) \in C(\beta) \) is convex if \( \beta \geq 1 \). It is natural now to raise the following question: What is the largest \( r_{\alpha, \beta} \), \( 0 < r_{\alpha, \beta} \leq 1 \) such that each \( f(z) \in S_*^\beta \) is a function in \( C(\beta) \) for \( |z| < r_{\alpha, \beta} \)? Again, Reade and Mocanu [4] have announced a sharp result for the general class \( S_*^\beta \).

**Theorem A (Reade and Mocanu).** If \( f(z) \in S_*^\beta \), then \( f(z) \in C(\beta) \) for \( |z| < r_{\alpha, \beta} = (1 + \beta + ((1 + \beta)^2 - 1)^{-1})^{1/2}, \beta \geq 0 \). This result is sharp for \( f(z) = z/(1 - z)^2 \).

We call \( r_{\alpha, \beta} \) the radius of \( \beta \)-convexity of the class \( S_*^\beta \). Here \( r_{0, \beta} = r_{\beta} \).

The object of this note is to extend Theorem A to the class \( S_*^\alpha \); in short to find \( r_{\alpha, \beta} \). In §2 a rough estimate of \( r_{\alpha, \beta} \) is given, Theorem 1. In §3, the number \( r_{\alpha, \beta} \) is completely determined, Theorem 2. The method used in §3 is that of V. A. Zmorović [7]. We also adopt his notations and thus we refer the reader to [7] for a deeper and perhaps a better understanding of §3.

2. Some estimate for \( r_{\alpha, \beta} \). Let \( P \) be the class of analytic functions in \( D \) such that if \( p(z) \in P \), \( p(0) = 1 \), and \( \text{Re}\{p(z)\} > 0 \) for all \( z \in D \). Let \( q(z) = zf'(z)/f(z) \), where \( f(z) \in S_*^\alpha \). Then there exists \( p(z) \in P \) such that

\[
q(z) = \alpha + (1 - \alpha)p(z) = (p(z) + h)/(1 + h),
\]

where \( h = \alpha/(1 - \alpha) \).

Using (3), (4) and the fact that

\[
1 + zf''(z)/f'(z) = q(z) + q'(z)/q(z),
\]

the radius of \( \beta \)-convexity of the class \( S_*^\alpha, r_{\alpha, \beta} \) becomes the smallest positive root of \( Q_{\alpha, \beta}(r) = 0 \), where

\[
Q_{\alpha, \beta}(r) = \min_{p \in P} \min_{|z| = r < 1} \text{Re}\{(1 - \alpha)(p(z) + h) + \beta qp'(z)/(p(z) + h)\}.
\]

Thus our problem is now reduced to finding the quantity

\[
Q(r) = \min_{p \in P} \min_{|z| = r < 1} \text{Re}\{\Psi(p(z), zp'(z))\},
\]

where \( \Psi(w, W) \) is an analytic function of the variables \( w \) and \( W \) in the \( W \)-plane and in the half plane \( \text{Re}\{w\} > 0 \). It is known [5] that the minimum
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in (6) is realized for functions of the form

\[
p(z) = \frac{1 + ze^{-i\theta_1}}{1 - ze^{-i\theta_1}} + \frac{1 + ze^{-i\theta_2}}{1 - ze^{-i\theta_2}},
\]

where \(\theta_1, \theta_2 \in [0, 2\pi]\), \(\lambda_1, \lambda_2 \geq 0\), and \(\lambda_1 + \lambda_2 = 1\).

Before determining \(Q(r)\), an elementary method yields a rough estimate for \(r_{\alpha, \beta}\). We need the following lemmas.

**Lemma 1.** If \(\phi(z)\) is analytic and \(|\phi(z)| \leq 1\) in \(D\), then

\[
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - r^2}
\]

for \(|z| = r < 1\).

Lemma 1 may be found in Carathéodory [1, p. 18].

**Lemma 2.** If \(q(z) = 1 + b_1 z + \cdots\), \(\Re\{q(z)\} > \alpha\), then

\[
\Re\{q(z)\} \geq \frac{1 - (1 - 2\alpha)r}{1 + r}
\]

valid for \(|z| = r < 1\).

This estimate readily follows from the fact that

\[
q(z) = \frac{(1 - (1 - 2\alpha)z\phi(z))}{1 + z\phi(z)},
\]

where \(\phi(z)\) is as in Lemma 1.

**Lemma 3.** If \(q(z)\) is as in Lemma 2, then

\[
|zq'(z)/q(z)| \leq 2(1 - \alpha)r/(1 - r)(1 + (1 - 2\alpha)r),
\]

for \(|z| = r < 1\).

**Proof.** From

\[
q(z) = \frac{(1 - (1 - 2\alpha)z\phi(z))}{1 + z\phi(z)},
\]

it follows that

\[
\frac{zq'(z)/q(z)}{1 + z\phi(z) - (1 - 2\alpha)z\phi(z)} = \frac{-2(1 - \alpha)(z^2\phi'(z) + z\phi(z))}{(1 + z\phi(z))(1 - (1 - 2\alpha)z\phi(z))}.
\]

The above may be written in the form

\[
zq'(z)/q(z) = -2(1 - \alpha)I_1(z)I_2(z),
\]

where

\[
I_1(z) = \frac{(z^2\phi'(z) + z\phi(z))}{1 - z^2\phi^2(z)},
\]

\[
I_2(z) = \frac{(1 - z\phi(z))}{1 - (1 - 2\alpha)z\phi(z)}.
\]

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Using the triangular inequality, the monotonicity of the right-hand side of $|I_1(z)|$ with respect to $|\phi'(z)|$ and on applying Lemma 1, we get

$$|I_1(z)| \leq \frac{2}{1 - r^2} \left( \frac{r^2(1 - |\phi(z)|^2) + r |\phi(z)| (1 - r^2)}{1 - r^2 |\phi(z)|^2} \right).$$

Let $0 \leq |\phi(z)| = t \leq 1$. The above inequality becomes

$$|I_1(z)| \leq g(t, r) = \frac{2}{1 - r^2} \left( \frac{r^2(1 - t^2) + rt(1 - r^2)}{1 - r^2 t^2} \right).$$

For fixed $r$,

$$\frac{dg}{dt} = 2r(1 - rt)^2/(1 - r^2 t^2)^2 = 0.$$

Therefore,

$$|I_1(z)| \leq \max_{0 \leq t \leq 1} g(t, r) = g(1, r) = 2r/(1 - r^2).$$

It is also clear that

$$|I_2(z)| \leq (1 + r)/(1 + (1 - 2a)r).$$

Using these estimates in (9) one gets (8).

**Theorem 1.** Let $f(z) = z + \alpha z^2 + \cdots$ be a function in $S^*$. Then $f(z)$ is $\beta$-convex in $|z| < R_{\alpha, \beta}$, where $R_{\alpha, \beta}$ is the smallest positive root of

$$(1 - 2\alpha)^2 r^3 - ((1 - 2\alpha)^2 + 2\beta(1 - \alpha)) r^2$$

$$- (1 + 2\beta(1 - \alpha)) r + 1 = 0. \quad (10)$$

**Proof.** Let $q(z) = zf''(z)/f(z)$. From (4),

$$\Re \{(1 - \alpha)(p + h) + \beta zp'(z)/(p(z) + h)\}$$

$$= \Re\{q(z) + \beta zq'(z)/q(z)\} \leq \Re\{q(z)\} - \beta |zq'(z)/q(z)|,$$

where $h = \alpha/(1 - \alpha), 0 \leq \alpha < 1$.

Applying Lemmas 2 and 3,

$$\Re\{(1 - \alpha)(p(z) + h) + \beta zp'(z)/(p(z) + h)\}$$

$$\leq [(1 - 2\alpha)^2 r^3 - ((1 - 2\alpha)^2 + 2\beta(1 - \alpha)) r^2 - (1 + 2\beta(1 - \alpha)) r + 1]$$

$$\leq (1 - r^2)(1 + (1 - 2\alpha)r).$$

From (5) and the above inequality each starlike function of order $\alpha$ in $D$ is $\beta$-convex in $|z| < R_{\alpha, \beta}$, where $R_{\alpha, \beta}$ is given by (10). Note that $r_{\alpha, \beta} \geq R_{\alpha, \beta}$ and if $\alpha = 0$, $r_{0, \beta} = R_{0, \beta} = r_{\beta}$ as given in Theorem A. Theorem 1, however, is not sharp since the estimates of Lemmas 2, 3 are sharp for

$$q_{\alpha}(z) = (1 - (1 - 2\alpha)z)/(1 + z)$$

but not at the same point. Indeed, $q_{\alpha}(z)$ realizes the estimate of Lemma 2.
at \( z = r \), while realizing the estimate of Lemma 3 at \( z = -r \). This is precisely the source of difficulties in such extremal problems.

3. The main theorem for \( r_{a,b} \). In this section we obtain \( r_{a,b} \) through an application of a theorem and technique due to V. A. Zmorovič [7] which is stated next.

**Theorem B (V. A. Zmorovič).** Let \( \Psi(w, W) = M(w) + N(w)W \), where \( M(w) \) and \( N(w) \) are defined and are finite in the half plane \( \text{Re}\{w\} > 0 \). Set

\[
\begin{align*}
w &= \lambda_1 \frac{1 + z_1^m}{1 - z_1^m} + \lambda_2 \frac{1 + z_2^m}{1 - z_2^m}, \\
W &= \lambda_1 \frac{2mz_1^m}{(1 - z_1^m)^2} + \lambda_2 \frac{2mz_2^m}{(1 - z_2^m)^2},
\end{align*}
\]

where \( z_1 \) and \( z_2 \) are any points on \( |z| = r < 1 \), \( \lambda_1 \geq 0 \), \( \lambda_2 \geq 0 \), \( \lambda_1 + \lambda_2 = 1 \). Then \( \Psi(w, W) \) can be put in the form

\[
\Psi(w, W) = M(w) + \frac{m}{2} (w^2 - 1)N(w) + \frac{m}{2} (\rho^2 - \rho_0^2)N(w)e^{2i\psi}
\]

where

\[
\left( 1 + z_k^m \right)/\left( 1 - z_k^m \right) = a + \rho e^{i\psi_k}, \quad (k = 1, 2),
\]

\[
w = a + \rho_0 e^{i\psi_0}, \quad 0 \leq \rho_0 \leq \rho,
\]

\[
a = \frac{1 + r_2^m}{1 - r_2^m}, \quad \rho = \frac{2r_1^m}{1 - r_2^m}, \quad e^{i\psi} = ie^{(i\psi_1 + \psi_2)/2}.
\]

Also

\[
\min \text{Re}\{\Psi(w, W)\} \equiv \Psi_\rho(w)
\]

\[
= \text{Re}\left\{ M(w) + \frac{m}{2} (w^2 - 1)N(w) \right\} - \frac{m}{2} |N(w)| \left( \rho^2 - \rho_0^2 \right).
\]

This minimum is reached when

\[
\exp[i(2\psi + \text{arg} N(w))] = -1.
\]

In our particular problem (5), \( m = 1 \),

\[
M(w) = (1 - \alpha)(w + h), \quad N(w) = \beta/(w + h).
\]

Thus from (6), (11) and the above relations,

\[
\min \text{Re}\{\Psi(w, W)\} \equiv \Psi_\rho(w)
\]

\[
= \text{Re}\left\{ (1 - \alpha)(w + h) + \frac{\beta}{2} \frac{w^2 - 1}{w + h} \right\} - \frac{\beta}{2} \frac{\rho^2 - \rho_0^2}{|w + h|}.
\]
The following remarks will be used later.

Remark 1. For a fixed \( w = a + \rho e^{i\phi_0}, \rho_0 < \rho \), and a suitably defined \( \psi_1 \) and \( \psi_2 \), a choice of \( \lambda_1 \) and \( \lambda_2 \) may be made, namely,

\[
\frac{\lambda_1}{\lambda_2} = \frac{|\rho e^{i\psi_2} - \rho_0 e^{i\psi_0}|}{|\rho e^{i\psi_1} - \rho_0 e^{i\psi_0}|}
\]

such that \( \Psi = (\Psi_1 + \Psi_2 + \pi)/2 \) becomes the angle of inclination of the secant through \( a + \rho e^{i\phi_0} \) and intersects the circle \( |w - a| = \rho \) at \( a + \rho e^{i\phi_k}, k = 1, 2 \). The choice of \( \frac{\lambda_1}{\lambda_2} \) is to maintain the correct relations between \( \rho_0 e^{i\psi_0}, \rho e^{i\psi_1} \) and \( \rho e^{i\psi_2} \) as required in the theorem. Thus \( \psi \) may assume any value in \([0, \pi]\).

Also as a consequence of formula (12), the minimum in (13) is reached when the point \( w, |w - a| < \rho \) is fixed and the secant through it, as described above, is perpendicular to \( e^{i\phi/2} \), where \( w + h = Re^{i\phi} \).

If we set \( w = a + \xi + i\eta, \rho_0^2 = \xi^2 + \eta^2 \leq \rho^2 \), then (13) becomes

\[
\Psi_\rho(w) \equiv \Psi_\rho(\xi, \eta) = \left( 1 + \frac{\beta}{2} - \alpha \right)(a + \xi + h) - \beta h
+ \frac{\beta}{2}(h^2 - 1)(a + \xi + h)R^{-2} - \frac{\beta}{2}(\rho^2 - \xi^2 - \eta^2)R^{-1},
\]

where \( R^2 = (a + \xi + h)^2 + \eta^2 \).

One can show that \( \partial \Psi_\rho / \partial \eta = (\beta/2) \eta R^{-1} S(\xi, \eta) \), where

\[
S(\xi; \eta) = [\xi^2 + 4(a + h)\xi + \rho^2 + \eta^2 + 2(a + h^2)]R
- [(h^2 - 1)(\xi + a + h)]
\geq [\xi^2 + 4(a + h)\xi + \rho^2 + 2(a + h^2) - 2(h^2 - 1)](\xi + a + h)
> 0,
\]

which shows that the minimum of \( \Psi_\rho(\xi, \eta) \) on every chord \( \xi \)-constant is reached when \( \eta = 0 \). Therefore, the minimum of \( \Psi_\rho(\xi, \eta) \) in the circle \( \xi^2 + \eta^2 \leq \rho^2 \) is reached on the diameter \( \eta = 0 \). Now set \( \eta = 0 \) and \( R = a + \xi + h \) in (1.4), we arrive at the following:

\[
\Psi_\rho(\xi, 0) \equiv l(R) = \left( 1 + \frac{\beta}{2} - \alpha \right)(a + \xi + h) - \beta h
+ \frac{\beta}{2}(h^2 - 1)(a + \xi + h)R^{-2} - \frac{\beta}{2}(\rho^2 - \xi^2)R^{-1}.
\]

From \( \xi = R - (a + h) \), \( \rho^2 = a^2 - 1 \),

\[
l(R) = (1 + \beta - \alpha)R + \beta(h^2 + ah)R^{-1} - \beta(a + 2h).
\]

Thus \( Q(r) = \min l(R), R \in [a + h - \rho, a + h + \rho]\), \( Q(r) \) is given by (6). Simple
calculations show that the absolute minimum of \( I(R) \) is realized at

\[
R_0 = \left( \frac{\beta (h^2 + ah)}{1 + \beta - \alpha} \right)^{1/2}
\]

Since

\[
R_0^2 = \frac{\beta (h^2 + ah)}{1 + \beta - \alpha} < h^2 + ah < (a + h + \rho)^2,
\]

\( R_0 < a + h + \rho \). However, \( R_0 \) may not be greater than \( a + h - \rho \). Therefore, if \( R_0 \not\in [a + h - \rho, a + h + \rho] \), then the minimum of \( I(R) \) is obtained at

\[
R_1 = a + h - \rho.
\]

The radius \( r_{a,\beta} \) is therefore determined either from

\[
Q(r) = \min I(R) = I(R_0) = 0,
\]

with \( R_0 \) given by (16), or from

\[
Q(r) = \min I(R) = I(R_1) = 0,
\]

with \( R_1 \) given by (17).

These two equations coincide for some \( \alpha_0 \) which will be determined later. Equations (18) and (19) may be written in the form, respectively,

\[
\beta a^2 - 4\alpha a - 4\alpha h = 0,
\]

\[
(1 - 3\alpha + \alpha h) r^2 - 2(1 + \beta - \alpha - \alpha h) r + 1 + \alpha + \alpha h = 0.
\]

It follows from (20) that

\[
r_1 = r_{a,\beta} = \sqrt{\frac{2\alpha - \beta + 2(\alpha^2 + \alpha h \beta)^{1/2}}{2\alpha + \beta + 2(\alpha^2 + \alpha h \beta)^{1/2}}}.
\]

Also from (21) follows that

\[
r_2 = r_{a,\beta} = \left[ (1 - 2\alpha + \beta (1 - \alpha)
+ ((1 - 2\alpha + \beta (1 - \alpha))^2 - (1 - 2\alpha)^2)^{1/2} \right]^{-1}.
\]

However, formula (23) cannot be used to determine \( r_{a,\beta} \) if

\[
\alpha \geq \frac{-\beta + (\beta^2 + 8\beta^{1/2})/4}{2},
\]

since \( r_2 \) would become greater than 1.

Also formula (22) cannot be used to determine \( r_{a,\beta} \) if

\[
\alpha \leq \beta/(4 + \beta),
\]

since \( r_1 \) would become a nonreal number.
To find $\alpha_0$ that makes the transition from (23) to (22), we set

$$r_1 = r_2,$$

and solve for $\alpha=\alpha_0$ where $\alpha_0$ is the smallest positive root of (26) which lies in

$$\left( \frac{\beta}{4 + \beta}, \frac{-\beta + (\beta^2 + 8\beta)^{1/2}}{4} \right).$$

Thus we have our main theorem.

**Theorem 2.** Let $\alpha_0$ be the smallest positive root of (26) which lies in the interval

$$\left( \frac{\beta}{4 + \beta}, \frac{-\beta + (\beta^2 + 8\beta)^{1/2}}{4} \right).$$

Then the radius of $\beta$-convexity for the class $S_*^\alpha$ is determined from (22) when $\alpha_0 \leq \alpha < 1$ and from (23) when $0 \leq \alpha \leq \alpha_0$.

Now we determine the extremal functions $f_0(z)$ for Theorem 2. Using Remark 1 and the fact that the minimum in case (22) is reached at a point on the diameter $\eta=0$ (not an endpoint) one gets $\psi_1 \equiv -\psi_2 \pmod{2\pi}$, and $\lambda_1/\lambda_2=1$. The extremal function given by (7) is therefore of the form

$$p(z) = \frac{1 + ze^{-i\theta}}{2} + \frac{1 + ze^{i\theta}}{2},$$

where $\theta$ is given by

$$R_0 = \text{Re}\{h + w\} = h + (1 - r_1^2)(1 - 2r_1 \cos \theta + r_1^2)^{-1},$$

$r_1$ is given by (22) and $R_0$ is given by (16). Hence the extremal function

$$f_0(z) = z(1 - 2z \cos \theta + z)^{-1 + \alpha}.$$  

In case of (22), the minimum is realized at an end point of the diameter $\eta=0$, thus $\psi_1 \equiv \psi_2 \pmod{2\pi}$. The function $p(z)$ of (7) has the form

$$p(z) = (1 + ze^{-i\theta})/(1 - ze^{-i\theta}),$$

or simply $p(z) = (1+z)/(1-z)$. Hence the extremal function

$$f_0(z) = z(1 - z)^{-2(1-\alpha)}.$$

**Remark 2.** (i) The case $\beta=1$ reduces $r_{\alpha, \beta}$ to be the radius of convexity of the class $S_*^\alpha$ which has been previously known for $\alpha=0$, $\alpha=1/2$. V. A. Zmorovič has provided the complete solution for such a case.

(ii) A. Schild [6] attempted to solve this problem ($\beta=1$) and actually succeeded if a certain condition is to be true. In fact he obtained $r_{\alpha,1}$ as
given by (22) and (23). Schild also calculated \( \alpha_0 = 0.335 \cdots \) and found the extremal functions (28) and (29) (for the case \( \beta = 1 \)).

Acknowledgement. The author acknowledges with thanks the valuable suggestions of Professor M. O. Reade of the University of Michigan.

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