PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, June 1973

SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

AN EXTENSION OF KOLMOGOROV'S THEOREM FOR CONTINUOUS COVARIANCES

G. D. ALLEN

ABSTRACT. The theorem of Kolmogorov stating that a nonnegative definite kernel on $N^1 \times N^1$ is the covariance of a stochastic process on N^1 is generalized to continuous nonnegative definite functions on $Y \times Y$, Y being a separable Hausdorff space. Also, a representation of such continuous nonnegative definite functions and their associated stochastic processes is provided.

1. In this note we provide a generalization of the result due to Kolmogorov [2] that if $\Gamma(n, m)$ is a nonnegative definite kernel from $N \times N$ to C, then there is a sequence $\{x_n | n \in N\}$ in a Hilbert space H such that $\Gamma(n, m) = (x_n, x_m)$. We are able to prove the theorem with the natural numbers N replaced by a separable Hausdorff space Y, H is a separable Hilbert space, and Γ continuous on $Y \times Y$. The proof is a direct Hilbert space construction, using no mappings. If the condition of continuity of Γ is dropped, then the separability of H must be dropped, as the example $\Gamma(t, t) = 1$, $\Gamma(t, t') = 0$, $t \neq t'$, given by R. M. Dudley shows. This problem was suggested to the author by P. Masani in February 1972.

We are also able to give a "canonical" representation of all continuous kernels of the nonnegative type.

2. In the following H is always a (complex) separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, and Y is always a separable Hausdorff space. We say that a function $\Gamma(t, t') \colon Y \times Y \to C$ is of the nonnegative type if for every finite sequences $\{t_i\} \subset Y$ and $\{\rho_i\} \subset C$ the sum

$$\sum_{i,j=1}^{n} \Gamma(t_i, t_j)\rho_i \bar{\rho}_j \geqq 0.$$

Received by the editors May 1, 1972 and, in revised form, September 1, 1972 and October 29, 1972.

AMS (MOS) subject classifications (1970). Primary 60G05; Secondary 60G30.

Key words and phrases. Representation of stochastic processes, covariance function, representation of covariance function.

(Here C denotes the complex numbers.) Furthermore, we say $\Gamma(t, t')$ is continuous if it is continuous in the usual product topology on $Y \times Y$ and hermitian if

$$\Gamma(t, t') = \overline{\Gamma(t', t)}.$$

With this notation we prove the

THEOREM. *Suppose $\Gamma: Y \times Y \to C$ is a continuous hermitian function of the nonnegative type. Then there is a continuous H-valued function $x(t)$ defined on Y such that $\Gamma(t, t') = (x(t), x(t'))$.*

PROOF. Let $S = \{t_i\}$ be a dense sequence in Y. For each positive integer n define the matrices $A_n = \{\Gamma(t_i, t_j), 1 \leqq i, j \leqq n\}$. These matrices are nonnegative definite and hence we can solve the matrix equations $C_n C_n^* = A_n$ for lower triangular nonnegative definite matrices C_n [**1**, p. 144]. By construction we can assume that C_n is equal to the upper left $n \times n$ submatrix of C_{n+m} for $m = 1, 2, \cdots$. So defined, the C_n are unique.

Now let $\{\phi_i\}$ be a complete orthonormal sequence for H. Define $(C_n = C_n(i, j), 1 \leqq i, j \leqq n)$

$$x_n(t_i) = \sum_{j=1}^{i} C_n(i, j)\phi_j.$$

By the above remarks we have $x_n(t_i) = x_m(t_i)$ for all t_i, $i = 1, \cdots, n$, if $m \geqq n$. Hence $\lim_{n\to\infty} x_n(t_i) = x(t_i)$ exists for each t_i. To establish continuity let $t'_i \to t' \in S$ where $\{t'_i\} \subset \{t_i\}$. Then

$$\|x(t') - x(t_i)\|^2 = \Gamma(t', t') - \Gamma(t', t_i) - \Gamma(t_i, t') + \Gamma(t_i, t_i),$$

and the term on the right-hand side tends to zero as $i \to \infty$. This proves that $x(t)$ is continuous on S. We now define $x(t)$ for all $t \in Y$ as the unique continuous extension to Y of $x(t)$ on S. It is clear that $(x(t), x(t')) = \Gamma(t, t')$, and this proves the theorem.

Define $\alpha_i(t) = (x(t), \phi_i)$. Then $\alpha_i(t_j) = 0, j < i$, $\alpha_i(t_i) \geqq 0$, and

$$x(t) = \sum_{i=1}^{\infty} \alpha_i(t)\phi_i.$$

By the continuity of $x(t)$ it follows that each $\alpha_i(t)$ is also continuous. Hence, the kernel $\Gamma(t, t')$ has the representation

$$(*) \qquad \Gamma(t, t') = \sum_{i=1}^{\infty} \alpha_i(t)\bar{\alpha}_i(t'),$$

and this representation is unique, relative to the sequence S, that is, there is only one sequence $\{\alpha_i(t)\}$ satisfying $\alpha_i(t_j) = 0, j < i$, $\alpha_i(t_i) = 0$. Furthermore the function $x(t)$ is unique in the same sense with the additional condition that the orthonormal (o.n.) sequence $\{\phi_i\}$ must be fixed. With a fixed o.n. sequence we call $x(t)$ and the associated covariance *canonical*, and the following corollary is obvious.

COROLLARY. (i) *Every continuous hermitian* $\Gamma(t, t')$ *of the nonnegative type has a representation of the type* (*).

(ii) *There is an isomorphism between canonical functions* $x(t)$ *and canonical covariances.*

(iii) *The canonical* $x(t)$ *and* $\Gamma(t, t')$ *form convex cones in the space* $C(Y; H)$ *and* $C(Y\times Y; C)$ *respectively.* $C(B; A)=\{$*continuous function* $x: B\to A\}$.

If Y is also compact we can impose the supremum norms on the functions $x(t)$ and the functions $\Gamma(t, t')$, namely

$$\|x(t)\|_H = \sup_{t\in Y} \|x(t)\| = \sup_{t\in Y} \left(\sum \alpha_i^2(t)\right)^{1/2},$$
$$\|\Gamma\|_Y = \sup_{s,t\in Y} \|\Gamma(s, t)\|.$$

It is easy to see that for nonnegative definite functions that $\|\Gamma\|_Y = \sup_{t\in Y} \|\Gamma(t, t)\|$, and we have the

COROLLARY. *If* Y *is compact there is an isometric isomorphism between the canonical* $x(t)$ *and canonical* $\Gamma(t, t')$.

REFERENCES

1. D. K. Faddeev and V. N. Faddeeva, *Computational methods in linear algebra*, Fizmatgiz, Moscow, 1960; English transl., Freeman, San Francisco, Calif., 1963. MR **28** #1742; #4659.

2. A. N. Kolmogorov, *Stationary sequences in Hilbert space*, Byull. Moskov. Gos. Univ. Mat. **2** (1941), no. 6, 1–40; English transl. by Natasha Artin. MR **5**, 101; MR **13**, 1138.

3. Ju. A. Rozanov, *Stationary random processes*, Fizmatgiz, Moscow, 1963; English transl., Holden-Day, San Francisco, Calif., 1967. MR **28** #2580; MR **35** #4985.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TEXAS 77843