SHORTER NOTES

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AN EXTENSION OF KOLMOGOROV'S THEOREM FOR CONTINUOUS COVARIANCES

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ABSTRACT. The theorem of Kolmogorov stating that a nonnegative definite kernel on $\mathbb{N} \times \mathbb{N}$ is the covariance of a stochastic process on $\mathbb{N}$ is generalized to continuous nonnegative definite functions on $Y \times Y$, $Y$ being a separable Hausdorff space. Also, a representation of such continuous nonnegative definite functions and their associated stochastic processes is provided.

1. In this note we provide a generalization of the result due to Kolmogorov [2] that if $\Gamma(n, m)$ is a nonnegative definite kernel from $\mathbb{N} \times \mathbb{N}$ to $C$, then there is a sequence $\{x_n|n \in N\}$ in a Hilbert space $H$ such that $\Gamma(n, m) = (x_n, x_m)$. We are able to prove the theorem with the natural numbers $N$ replaced by a separable Hausdorff space $Y$, $H$ is a separable Hilbert space, and $\Gamma$ continuous on $Y \times Y$. The proof is a direct Hilbert space construction, using no mappings. If the condition of continuity of $\Gamma$ is dropped, then the separability of $H$ must be dropped, as the example $\Gamma(t, t) = 1$, $\Gamma(t, t') = 0$, $t \neq t'$, given by R. M. Dudley shows. This problem was suggested to the author by P. Masani in February 1972.

We are also able to give a “canonical” representation of all continuous kernels of the nonnegative type.

2. In the following $H$ is always a (complex) separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$, and $Y$ is always a separable Hausdorff space. We say that a function $F(t, t'): Y \times Y \to C$ is of the nonnegative type if for every finite sequences $\{t_i\} \subset Y$ and $\{\rho_i\} \subset C$ the sum

$$\sum_{i, j=1}^{n} \Gamma(t_i, t_j) \rho_i \rho_j \geq 0.$$
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(Here $C$ denotes the complex numbers.) Furthermore, we say $\Gamma(t, t')$ is continuous if it is continuous in the usual product topology on $Y \times Y$ and hermitian if

$$\Gamma(t, t') = \overline{\Gamma(t', t)}.$$ 

With this notation we prove the

**Theorem.** Suppose $\Gamma: Y \times Y \to C$ is a continuous hermitian function of the nonnegative type. Then there is a continuous $H$-valued function $x(t)$ defined on $Y$ such that $\Gamma(t, t') = (x(t), x(t'))$.

**Proof.** Let $S = \{t_i\}$ be a dense sequence in $Y$. For each positive integer $n$ define the matrices $A_n = (\Gamma(t_i, t_j), 1 \leq i, j \leq n)$. These matrices are non-negative definite and hence we can solve the matrix equations $C_n C_n^* = A_n$ for lower triangular nonnegative definite matrices $C_n$ [1, p. 144]. By construction we can assume that $C_n$ is equal to the upper left $n \times n$ submatrix of $C_{n+m}$ for $m = 1, 2, \cdots$. So defined, the $C_n$ are unique.

Now let $\{\phi_j\}$ be a complete orthonormal sequence for $H$. Define ($C_n = C_n(i, j), 1 \leq i, j \leq n$)

$$x_n(t_i) = \sum_{j=1}^{n} C_n(i, j) \phi_j.$$ 

By the above remarks we have $x_n(t_i) = x_m(t_i)$ for all $t_i, i = 1, \cdots, n$, if $m \geq n$. Hence $\lim_{n \to \infty} x_n(t_i) = x(t_i)$ exists for each $t_i$. To establish continuity let $t'_i \to t' \in S$ where $\{t'_i\} \subseteq \{t_i\}$. Then

$$\|x(t') - x(t_i)\|^2 = \Gamma(t', t') - \Gamma(t', t_i) - \Gamma(t_i, t') + \Gamma(t_i, t_i),$$

and the term on the right-hand side tends to zero as $i \to \infty$. This proves that $x(t)$ is continuous on $S$. We now define $x(t)$ for all $t \in Y$ as the unique continuous extension to $Y$ of $x(t)$ on $S$. It is clear that $(x(t), x(t')) = \Gamma(t, t')$, and this proves the theorem.

Define $\alpha_i(t) = (x(t), \phi_i)$. Then $\alpha_i(t_j) = 0, j < i, \alpha_i(t_i) \geq 0$, and

$$x(t) = \sum_{i=1}^{\infty} \alpha_i(t) \phi_i.$$ 

By the continuity of $x(t)$ it follows that each $\alpha_i(t)$ is also continuous. Hence, the kernel $\Gamma(t, t')$ has the representation

$$\Gamma(t, t') = \sum_{i=1}^{\infty} \alpha_i(t) \overline{\alpha}_i(t'),$$

and this representation is unique, relative to the sequence $S$, that is, there is only one sequence $\{\alpha_i(t)\}$ satisfying $\alpha_i(t_j) = 0, j < i, \alpha_i(t_i) = 0$. Furthermore the function $x(t)$ is unique in the same sense with the additional
condition that the orthonormal (o.n.) sequence \( \{\phi_i\} \) must be fixed. With a fixed o.n. sequence we call \( x(t) \) and the associated covariance canonical, and the following corollary is obvious.

**Corollary.** (i) *Every continuous hermitian \( \Gamma(t, t') \) of the nonnegative type has a representation of the type (*)*.  
(ii) *There is an isomorphism between canonical functions \( x(t) \) and canonical covariances.*  
(iii) *The canonical \( x(t) \) and \( \Gamma(t, t') \) form convex cones in the space \( C(Y; H) \) and \( C(Y \times Y; C) \) respectively. \( C(B; A) = \{ \text{continuous function } x : B \rightarrow A \} \).*

If \( Y \) is also compact we can impose the supremum norms on the functions \( x(t) \) and the functions \( \Gamma(t, t') \), namely

\[
\|x(t)\|_H = \sup_{t \in Y} \|x(t)\| = \sup_{t \in Y} (\sum \alpha_n^2(t))^{1/2}, \\
\|\Gamma\|_Y = \sup_{s, t \in Y} \|\Gamma(s, t)\|.
\]

It is easy to see that for nonnegative definite functions that \( \|\Gamma\|_Y = \sup_{t \in Y} \|\Gamma(t, t)\| \), and we have the

**Corollary.** *If \( Y \) is compact there is an isometric isomorphism between the canonical \( x(t) \) and canonical \( \Gamma(t, t') \).*

**References**


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