SHORTER NOTES

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AN EXTENSION OF KOLMOGOROV'S THEOREM FOR CONTINUOUS COVARIANCES

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Abstract. The theorem of Kolmogorov stating that a non-negative definite kernel on \( N \times N \) is the covariance of a stochastic process on \( N \) is generalized to continuous nonnegative definite functions on \( Y \times Y \), \( Y \) being a separable Hausdorff space. Also, a representation of such continuous nonnegative definite functions and their associated stochastic processes is provided.

1. In this note we provide a generalization of the result due to Kolmogorov \([2]\) that if \( \Gamma(n, m) \) is a nonnegative definite kernel from \( N \times N \) to \( C \), then there is a sequence \( \{x_n|n \in N\} \) in a Hilbert space \( H \) such that \( \Gamma(n, m) = (x_n, x_m) \). We are able to prove the theorem with the natural numbers \( N \) replaced by a separable Hausdorff space \( Y \), \( H \) is a separable Hilbert space, and \( \Gamma \) continuous on \( Y \times Y \). The proof is a direct Hilbert space construction, using no mappings. If the condition of continuity of \( \Gamma \) is dropped, then the separability of \( H \) must be dropped, as the example \( \Gamma(t, t') = 1, \Gamma(t, t') = 0, t \neq t' \), given by R. M. Dudley shows. This problem was suggested to the author by P. Masani in February 1972.

We are also able to give a "canonical" representation of all continuous kernels of the nonnegative type.

2. In the following \( H \) is always a (complex) separable Hilbert space with inner product \((\cdot, \cdot)\) and norm \(\|\cdot\|\), and \( Y \) is always a separable Hausdorff space. We say that a function \( \Gamma(t, t'): Y \times Y \to C \) is of the nonnegative type if for every finite sequences \( \{t_i\} \subseteq Y \) and \( \{\rho_i\} \subseteq C \) the sum

\[
\sum_{i, j=1}^{n} \Gamma(t_i, t_j) \rho_i \bar{\rho}_j \geq 0.
\]
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(Here $C$ denotes the complex numbers.) Furthermore, we say $\Gamma(t, t')$ is continuous if it is continuous in the usual product topology on $Y \times Y$ and hermitian if

$$\Gamma(t, t') = \overline{\Gamma(t', t)}.$$ 

With this notation we prove the

**Theorem.** Suppose $\Gamma: Y \times Y \to C$ is a continuous hermitian function of the nonnegative type. Then there is a continuous $H$-valued function $x(t)$ defined on $Y$ such that $\Gamma(t, t') = (x(t), x(t'))$.

**Proof.** Let $S = \{t_i\}$ be a dense sequence in $Y$. For each positive integer $n$ define the matrices $A_n = \{\Gamma(t_i, t_j), 1 \leq i, j \leq n\}$. These matrices are nonnegative definite and hence we can solve the matrix equations $C_n C_n^* = A_n$ for lower triangular nonnegative definite matrices $C_n$ [1, p. 144]. By construction we can assume that $C_n$ is equal to the upper left $n \times n$ submatrix of $C_{n+m}$ for $m = 1, 2, \ldots$. So defined, the $C_n$ are unique.

Now let $\{\phi_i\}$ be a complete orthonormal sequence for $H$. Define

$$(C_n = C_n(i, j), 1 \leq i, j \leq n)$$

$$x_n(t_i) = \sum_{j=1}^{i} C_n(i, j) \phi_j.$$ 

By the above remarks we have $x_n(t_i) = x_m(t_i)$ for all $t_i, i = 1, \ldots, n$, if $m \geq n$. Hence $\lim_{n \to \infty} x_n(t_i) = x(t_i)$ exists for each $t_i$. To establish continuity let $t_i \to t' \in S$ where $\{t_i\} \subset \{t_j\}$. Then

$$\|x(t') - x(t_i)\|^2 = \Gamma(t', t') - \Gamma(t', t_i) - \Gamma(t_i, t') + \Gamma(t_i, t_i),$$

and the term on the right-hand side tends to zero as $i \to \infty$. This proves that $x(t)$ is continuous on $S$. We now define $x(t)$ for all $t \in Y$ as the unique continuous extension to $Y$ of $x(t)$ on $S$. It is clear that $(x(t), x(t')) = \Gamma(t, t')$, and this proves the theorem.

Define $\alpha_i(t) = (x(t), \phi_i)$. Then $\alpha_i(t_j) = 0, j < i, \alpha_i(t_i) \geq 0$, and

$$x(t) = \sum_{i=1}^{\infty} \alpha_i(t) \phi_i.$$ 

By the continuity of $x(t)$ it follows that each $\alpha_i(t)$ is also continuous. Hence, the kernel $\Gamma(t, t')$ has the representation

$$(*) \quad \Gamma(t, t') = \sum_{i=1}^{\infty} \alpha_i(t) \overline{\alpha_i(t')},$$

and this representation is unique, relative to the sequence $S$, that is, there is only one sequence $\{\alpha_i(t)\}$ satisfying $\alpha_i(t_j) = 0, j < i, \alpha_i(t_i) = 0$. Furthermore the function $x(t)$ is unique in the same sense with the additional
condition that the orthonormal (o.n.) sequence \{\phi_i\} must be fixed. With a fixed o.n. sequence we call \(x(t)\) and the associated covariance canonical, and the following corollary is obvious.

**Corollary.**

(i) Every continuous hermitian \(\Gamma(t, t')\) of the nonnegative type has a representation of the type (*)

(ii) There is an isomorphism between canonical functions \(x(t)\) and canonical covariances.

(iii) The canonical \(x(t)\) and \(\Gamma(t, t')\) form convex cones in the space \(C(Y; H)\) and \(C(Y \times Y; C)\) respectively. \(C(B; A) = \{\text{continuous function } x: B \to A\}\).

If \(Y\) is also compact we can impose the supremum norms on the functions \(x(t)\) and the functions \(\Gamma(t, t')\), namely

\[
\|x(t)\|_H = \sup_{t \in Y} \|x(t)\| = \sup_{t \in Y} \left(\sum_{i} \phi_i^2(t)\right)^{1/2},
\]

\[
\|\Gamma\|_Y = \sup_{s, t \in Y} \|\Gamma(s, t)\|.
\]

It is easy to see that for nonnegative definite functions that \(\|\Gamma\|_Y = \sup_{t \in Y} \|\Gamma(t, t)\|\), and we have the

**Corollary.** If \(Y\) is compact there is an isometric isomorphism between the canonical \(x(t)\) and canonical \(\Gamma(t, t')\).

**REFERENCES**


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