A NOTE ON TORSION-FREE ABELIAN GROUPS OF FINITE RANK

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Abstract. Let G be a torsion-free abelian group of rank n and X={x_1, \cdots, x_n} a maximal set of rationally independent elements in G. It is well known that any g \in G can be uniquely written g = \alpha_1 x_1 + \cdots + \alpha_n x_n, for some \alpha_1, \cdots, \alpha_n \in \mathbb{Q}, the rational numbers. This enables us to define, for any such (G, X), a collection of subgroups of \mathbb{Q} and "natural" isomorphisms, denoted by S(G, X). It is known that if G is of rank two, then G may be recovered from S(G, X) in a natural way. The following result is obtained for groups of rank greater than two:

Theorem. Let G, G' be torsion free abelian groups of finite rank with S(G, X) = S(G', X') for suitable X, X'. Let F, F' be the free subgroups of G, G' generated by X, X'. Then G/F \cong G'/F'.

An additional condition is given for pairs (G, X), (G', X') such that S(G, X) = S(G', X') implies G \cong G'.

1. Schemes of groups. Let G be a group^1 of rank n and X={x_1, \cdots, x_n} a maximal set of rationally independent elements in G. It is well known that any g \in G can be uniquely written g = \alpha_1 x_1 + \cdots + \alpha_n x_n, for some \alpha_1, \cdots, \alpha_n \in \mathbb{Q}, the rational numbers. This enables us to define, for any such (G, X), a collection of subgroups of \mathbb{Q} as follows:

Definition 1. Let 1 \leq i \leq n and i \notin J \subseteq \{1, 2, \cdots, n\}. Then

\[ A^i_J = A^i_J(G, X) \]

\[ \{ \alpha \in \mathbb{Q} \mid \alpha_1 x_1 + \cdots + \alpha_i-1 x_{i-1} + \alpha_i x_i + \alpha_{i+1} x_{i+1} + \cdots + \alpha_n x_n \in G \text{ for some set of } \alpha_j \in \mathbb{Q} \text{ with } \alpha_j = 0 \; \forall j \notin J \} \]

The A^i_J are clearly subgroups of the additive rationals and contain the integers, Z. Furthermore, A^i_J \supseteq A^i_J' if J \subseteq J'. For convenience, we write A^i = A^i_J. The A^i_J can be used to define a collection of "natural" isomorphisms.

Lemma 1. Given 1 \leq i \neq j \leq n, with i, j \notin J \subseteq \{1, 2, \cdots, n\}, there exists an isomorphism

\[ A^i_J / A^i_J (i \cup J) \cong A^j_J / A^j_J (i \cup J) \]

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^1 We write "group" for torsion-free abelian group throughout.

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Proof. Let $a_i \in A^i_J$ and write $\sum_{k=1}^{n} a_k x_k \in G$ for some choice of $a_k \in Q$, $k \neq i$, with $a_k = 0$ for $k \in J$. Then $(a_i + A^1_{(i) \cup J})^\theta = a_J + A^1_{(i) \cup J}$ is the required isomorphism.

We call the collection of groups $A^i_J$ and isomorphisms $\theta$ the scheme of $(G, X)$, and denote it by $S(G, X)$.

Lemma 2. Given $1 \leq i, j, k \leq n$ with $i, j, k \notin J \subset \{1, \cdots, n\}$ and $i, j, k$ distinct, then

$$\begin{array}{c}
A^i_{J}/A^i_{(i) \cup J} \xrightarrow{\theta} A^j_{J}/A^i_{(i) \cup J} \\
\uparrow NAT \quad \uparrow NAT \\
A^i_{[k] \cup J}/A^i_{[k,j] \cup J} \xrightarrow{\theta} A^j_{[k] \cup J}/A^i_{[k,j] \cup J}
\end{array}$$

is a commutative diagram where the $\theta$'s are the isomorphisms of Lemma 1 and $NAT$ is the composition of the obvious inclusion and factor maps.

Proof. The proof follows immediately from the definition of the maps $\theta$.

If $G$ has rank 2, Beaumont and Wisner [2] showed that $G$ may be recovered from the groups $A^i_J$ and isomorphism $\theta$ in the following way: Let $x_1, x_2$ be independent elements in $G$ and $(A^1_1, A^2_1, A^2_2, A^1_3, \theta)$ the scheme of $(G, \{x_1, x_2\})$. Then $G' = \{(x, \beta) | x \in A_1, (\alpha + A^2_2)^\theta = \beta + A^1_3\}$ is isomorphic to $G$. Moreover, $S(G', \{(1, 0), (0, 1)\}) = S(G, \{x_1, x_2\})$.

The question then arises: To what extent does $S(G, X)$ determine $G$ in general? The results which follow give a partial answer.

Since any group $G$ of rank $n$ with maximal independent set $X = \{x_1, \cdots, x_n\}$ can be identified with a subgroup of $Q^n$ under the map $\alpha x_1 + \cdots + \alpha_n x_n \rightarrow (\alpha_1, \cdots, \alpha_n)$, it suffices to consider pairs $(G, X)$ with $Z^n \leq G \leq Q^n$ and $X = \{(1, 0, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, 0, \cdots, 0, 1)\}$. Let $(G, X)$ and $(G', X)$ be two such pairs.

Lemma 3. If $S(G, X) = S(G', X)$ then given any element $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in G$ and fixed integers $i \neq j$, with $1 \leq i, \leq j \leq n$, there exists an element $(\alpha'_1, \alpha'_2, \cdots, \alpha'_n) \in G'$ with $\alpha_i = \alpha'_i, \alpha_j = \alpha'_j$. Moreover, it is possible to choose $\alpha'_k = 0$ whenever $\alpha_k = 0$.

Proof. By symmetry, we may take $i = 1, j = 2$. Let $K = \{k | \alpha_k = 0\}$. Since $\alpha_1 \in A^1_K$ and $A^1_K(G, X) = A^1_K(G', X)$, there exists $(\alpha_1, \beta_2, \beta_3, \cdots, \beta_n) \in G'$ with $\beta_k = 0$ for $k \in K$. Thus, $\beta_2 + A^2_{1 \cup K} = [\alpha_1 + A^1_{2 \cup K}]^\theta = \alpha_2 + A^3_{1 \cup K}$. Hence, $\alpha_2 - \beta_2 \in A^2_{1 \cup K}$, and there exists $(0, \alpha_3 - \beta_3, \gamma_3, \cdots, \gamma_n) \in G'$ with $\gamma_k = 0$ for $k \in K$. Then $(\alpha_1, \alpha_2, \cdots, \alpha_n) = (0, \alpha_3 - \beta_2, \gamma_3, \cdots, \gamma_n) + (\alpha_1, \beta_2, \cdots, \beta_n) \in G'$ with $\alpha'_k = 0$ for $k \in K$. 

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Lemma 4. Let \( G, G', X \) be as in Lemma 3, and \( p \) a prime integer. Let \( g = (a_1, \ldots, a_n) \in G \) and \( g'_1, \ldots, g'_t \in G' \) be such that \( g = g + Z^n, \ g'_1 = g_1 + Z^n, \ldots, g'_t = g_t + Z^n \) are independent elements in the \( p \)-primary component of \( Q^n \). Then there exists \( g' = (a'_1, \ldots, a'_n) \in G' \), with \( \text{order}(g') \geq \text{order}(g) = p^k \), such that \( g_1, \ldots, g'_t, g' \) are independent. If \( a_r = a_r/p^k, a'_s = a'_s/p^k \), then \( g' \) can be chosen so that \( a_r = a'_r, a'_s = a'_s \), where \( c \) is an integer with \( (c, p) = 1 \).

Furthermore, we can choose \( a'_i = 0 \) whenever \( a_i = 0 \).

Proof. We proceed by induction on the number, \( m \), of nonzero coordinates of \( g \). If \( m = 1, 2 \), then \( g \in G' \) by Lemma 3 and the result follows.

Suppose now that \( g = (a_1, \ldots, a_m, 0, \ldots, 0) \). We assume, without loss of generality, that all integral components of \( g \) are zero, and, by renumbering if necessary, that the nonzero coordinates appear first.

Write \( a_i = a_i/p^k, 1 \leq k_i \leq k \), where \( (a_i, p) = 1 \). Again by renumbering, assume \( r = 1, s = 2 \). Thus, \( k_1 = k \). By Lemma 3, there exists

\[
h_1 = (a_1/p^k, a_2/p^k, \beta_3, \ldots, \beta_m, 0, \ldots, 0) \in G'.
\]

Choose an integer \( c \) with \( (c, p) = 1 \) such that

\[
h_2 = ch_1 = (c a_1/p^k, c a_2/p^k, b_3/p^k, \ldots, b_m/p^k, 0, \ldots, 0).
\]

Note that \( \{g_1', \ldots, g_t', c g\} \) is still independent. (The integer \( c \) times a power of \( p \) is divisible by the denominators of \( \beta_3 \cdots \beta_m \).)

Case I. Suppose \( l_i = k_i \) and \( p | c a_i - b_i \) for \( 3 \leq i \leq m \). Then \( \text{order}(h_2) = p^k \).

We show \( \{g_1', \ldots, g_t', h_2\} \) is independent. It suffices to show \( \{p^{i-1}g_1', \ldots, p^{i-1}g_t', p^{k-1}h_2\} \) is independent in \( G'[p] \), where order \( g_i' = p^i \). Suppose

\[
u_i p^i g_1' + \cdots + u_t p^t g_t' + u p^{k-1} h_2 = 0
\]

with \( (p, u) = 1 \). Clearly, \( u_i p^i g_i' + \cdots + u_t p^t g_t' + u p^{k-1} c g \neq 0 \). Subtraction yields \( u p^{k-1}(c g - h_2) \neq 0 \). But, since \( p | c a_i - b_i \), \( i = 3, 4, \ldots, m \), \( u p^{k-1}(c g - h_2) = 0 \), a contradiction. Thus, in Case I, \( g' = h_2 \) is the desired element of \( G' \).

Case II. Assume without loss of generality that \( l_3 \neq k_3 \), or that \( p | c a_3 - b_3 \). It then is easy to show that \( 1/p^{k_3} \in A'_{1, m+1, m+2, \ldots, n} \). Thus there exist \( \Gamma_1 = (0, x, c a_3/p^{k_3}, x_4, \ldots, x_m, 0, 0, \ldots, 0) \in G \) and (by Lemma 3)

\[
\Gamma_2 = (0, c a_3/p^{k_3}, x_4, \ldots, x_m, 0, 0, \ldots, 0) \in G'.
\]

Applying Lemma 3 again to \( \Gamma_2 = c g - \Gamma_1 \), we have

\[
\Gamma_2' = (c a_3/p^{k_3}, c a_3/p^{k_3} - x, 0, y_4', \ldots, y_m', 0, 0, \ldots, 0) \in G'.
\]

Consider \( h_2 - \Gamma_1' - \Gamma_2' = (0, b_3/p^{k_3} - c a_3/p^{k_3}, z_4', \ldots, z_m', 0, 0, \ldots, 0) \).

Since \( l_3 \neq k_3 \) or \( p | b_3 - c a_3 \), it follows that \( 1/p^{k_3} \in A'_{1, m+1, m+2, \ldots, n} \). Let

\[
\Gamma_3 = (0, 0, c a_3/p^{k_3}, v_4, \ldots, v_m, 0, 0, \ldots, 0) \in G
\]

and

\[
\Gamma_3' = (0, 0, c a_3/p^{k_3}, v_4', \ldots, v_m', 0, 0, \ldots, 0) \in G'.
\]
Finally, let
\[ \Gamma'_4 = \Gamma'_1 + \Gamma'_2 - \Gamma'_3 = (ca_1/p^{k_1}, ca_2/p^{k_2}, 0, w'_4, \ldots, w'_m, 0, 0, \ldots, 0). \]

If \( \{\tilde{g}_1, \ldots, \tilde{g}_t, \Gamma'_4\} \) is independent, it is easy to construct the desired element \( g' \in G' \). If \( \{\tilde{g}_1, \ldots, \tilde{g}_t, c\Gamma'_3\} \) is independent, the result follows from the induction hypothesis, after multiplication by a suitable integer prime to \( p \).

Otherwise, \( \Gamma'_4 \) is dependent on, and \( \Gamma'_3 \) is independent of the set \( \{\tilde{g}_1, \ldots, \tilde{g}_t\} \). Multiplying \( \Gamma'_3 \) by a suitable integer \( d \), prime to \( p \), and applying the induction hypothesis, we obtain
\[ \Gamma''_3 = (0, 0, cda_0/p^{k_3}, v''_4, \ldots, v''_m, 0, 0, \ldots, 0) \in G' \]

independent of \( \tilde{g}_1, \ldots, \tilde{g}_t \). Now \( \Gamma''_3 + d\Gamma'_4 \) is still independent, and we can choose \( e \) prime to \( p \) such that \( g' = e(\Gamma''_3 + d\Gamma'_4) \) is the desired element.

**Lemma 5.** Let \( G, G' \), \( X \) be as in Lemma 3. Let \( \tilde{g}_1, \ldots, \tilde{g}_t \) be independent elements in the \( p \)-primary component of \( G \). Then there exist \( \tilde{g}_1, \ldots, \tilde{g}_t \) independent in the \( p \)-primary component of \( G' \) with order(\( \tilde{g}_i \)) \( \leq \) order(\( \tilde{g}_i \)).

**Proof.** The result follows by using induction on \( t \) and applying Lemma 4. We now prove the main theorem.

**Theorem A.** Let \( G, G' \) be groups of rank \( n \) with maximal independent sets \( Y, Y' \) respectively, such that \( S(G, Y) = S(G', Y') \). Let \( F, F' \) be the free subgroups of \( G, G' \) generated by \( Y, Y' \) respectively. Then \( G/F \cong G'/F' \).

**Proof.** As before, we identify \( G \) and \( G' \) with subgroups of \( \mathbb{Z}^n \) containing \( \mathbb{Z}^n \), such that \( Y \) and \( Y' \) are identified with the set \( X \) of standard basis vectors. Write
\[
G/Z^n = \bigoplus A_p, \quad G'/Z^n = \bigoplus A'_p
\]
where \( A_p \) and \( A'_p \) are the respective \( p \)-primary components. Note that \( A_p \) and \( A'_p \) are subgroups of \((\mathbb{Q}^n/Z^n)_p\), and therefore
\[
A_p = \bigoplus \sum_{i=1}^{r} \mathbb{Z}(p^{k_i}),
\]
\[
A'_p = \bigoplus \sum_{i=1}^{r'} \mathbb{Z}(p^{k'_i}) \quad \text{where} \quad 1 \leq k_i \leq \infty, \quad 1 \leq k'_i \leq \infty.
\]

Applying Lemma 5, we have that \( A_p \cong A'_p \) for all \( p \).

Note that the converse to Theorem A is false. It is easy to construct nonisomorphic subgroups \( G \) and \( G' \) of \( \mathbb{Q} \oplus \mathbb{Q} \) which contain \( \mathbb{Z} \oplus \mathbb{Z} \) with \( G/Z \cong G'/Z \). Since groups of rank two are determined up to isomorphism by their schemes, \( S(G, X) \neq S(G', X') \) for any \( X, X' \).
2. The intersection property. In this section we give a condition under which a group may be recovered from its scheme, \( S(G, X) \).

Definition 2. The groups \( A_j \) associated with \( (G, X) \) are said to satisfy the intersection property iff \( A_j \cap A_j' = A_j \cap A_j' \) for all \( 1 \leq i \leq n \) and \( i \notin J \cup J' \), where \( J \) and \( J' \) are subsets of \( \{1, \ldots, n\} \). In this case we will also say \( (G, X) \) satisfies the intersection property.

Remarks. It is immediate that if \( G \) has rank \( \leq 2 \), \( (G, X) \) satisfies the intersection property for any maximal independent set \( X \subseteq G \). If \( (G_i, X_i) \) has the intersection property for \( i = 1, 2, \ldots, n \), then \( \bigcup_{i=1}^{n} X_i \) has the intersection property, where \( X_i \) denotes the natural embedding of \( X_i \) into the direct sum. Let \( v = \sum_{k=0}^{\infty} v_k p^k \), \( w = \sum_{k=0}^{\infty} w_k p^k \) be \( p \)-adic units such that \( 1, v, w \) are rationally independent. Let \( V_0 = W_0 = 0 \), \( V_i = \sum_{k=0}^{i-1} v_k p^k \), \( W_i = \sum_{k=0}^{i-1} w_k p^k \), for \( i > 1 \). It can be shown using the methods of [1] that

\[
G = \{(r/p^k, r/p^k V_k + t, r/p^k W_k + l) | (r, p) = 1, t, l \in \mathbb{Z}\}
\]

is a strongly indecomposable rank 3 group; and it is easy to verify directly that \( (G, X) \) has the intersection property with \( X = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \).

Theorem B. Let \( (G, X) \) satisfy the intersection property for \( G \) a group of rank \( n \) and \( X = \{x_1, \ldots, x_n\} \) a maximal independent subset. Let \( S(G, X) = \{A_j \} \). Then

\[
G = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n | (x_1, \ldots, x_n) \in G \}
\]

is isomorphic to \( G \) under the map \( (x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i x_i \).

Proof. We show that \( (x_1, \ldots, x_n) \in G \) iff \( \sum_{i=1}^{n} x_i x_i \in G \). Clearly, by definition of the maps \( \theta \), \( \sum_{i=1}^{n} x_i x_i \in G \) implies \( (x_1, \ldots, x_n) \in G \). Let \( (x_1, \ldots, x_n) \in G \). Then \( x_1 \in A_1 \), so \( g_1 = x_1 + x_2 x_3 + \cdots + x_n x_n \in G \) for some \( x_2, \ldots, x_n \). But \( x_2 + A_2 = (x_1 + A_2)^{\theta} = x_1 + A_2 \), so \( x_2 \in A_2 \). Thus, there exists \( g_2 = (x_2 - x_3) x_3 + x_3 x_3 + \cdots + x_n x_n \in G \). Now \( g_1 + g_2 = x_1 x_1 + x_2 x_2 + (x_3 + x_3) x_3 + \cdots + (x_n + x_n) x_n \in G \). This yields \( x_3 - (g_3 + x_3^\prime) \in A_3 \cap A_3^\prime \). Since \( (G, X) \) satisfies the intersection property, we have \( x_3 - (g_3 + x_3^\prime) \in A_3 \), and there exists \( g_3 = [x_3 - (g_3 + x_3^\prime)] x_3 + x_3^\prime x_3^\prime + \cdots + x_n x_n \in G \). We continue in this way, obtaining \( g_1, \ldots, g_n \), such that \( \sum_{i=1}^{n} x_i x_i = g_1 + g_2 + \cdots + g_n \in G \).

Corollary. Let \( (G, X) \) satisfy the intersection property, and let \( S(G, X) = S(G', X') \). Then \( G \cong G' \).

Remark (Added in proof). Theorem B and the above Corollary do not hold in general. There exist two nonisomorphic groups of rank 3 which have the same schemes with respect to the standard basis.
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