PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 39, Number 1, June 1973

ADDITIVITY AND INDEFINITE INTEGRATION FOR McSHANE'S P-INTEGRAL[1]

C. H. SCANLON

ABSTRACT. Suppose $[a, b]$ is a closed real number interval, $A=\{(p, q]; (p, q]\subset(a, b]\}$, and U is a real valued function on $[a, b]\times A$. For $c\in(a, b)$, necessary and sufficient conditions are given for P-integrability of U on $(a, c]$ and $(c, b]$ to imply P-integrability on $(a, b]$. Suppose U is P-integrable on $(a, b]$ and $g(x)=P\int_a^x U$ for each $x\in[a, b]$. Necessary and sufficient conditions are given for g to be respectively continuous, bounded, and of bounded variation.

1. **Preliminaries.** Suppose $a=t_0<t_1<\cdots<t_n=b$ and $x_1, x_2, \cdots, x_n\in[a, b]$. Then $\Pi=\{(x_1, (t_0, t_1]), \cdots, (x_n, (t_{n-1}, t_n])\}$ is called a P-partition of $[a, b]$. If δ is a positive real valued function on $[a, b]$, then the statement that Π is δ-fine means that $(t_{i-1}, t_i)\subset(x_i-\delta(x_i), x_i+\delta(x_i))$ for $i=1, \cdots, n$.

The statement that U is P-integrable on $(a, b]$ means that there is a real number denoted by $P\int_a^b U$ such that if $\varepsilon>0$ there is a positive real valued function δ on $[a, b]$ such that if $\Pi=\{(x_i, (t_{i-1}, t_i])\}_{i=1}^n$ is any δ-fine P-partition of $(a, b]$ then

$$\left|\sum_{i=1}^n U(x_i, (t_{i-1}, t_i]) - P\int_a^b U\right| < \varepsilon.$$

Variations in the following lemma can be found in ([**1**], [**2**], [**3**]).

LEMMA K. *If U is P-integrable on $[a, b]$ and $\varepsilon>0$, there is a positive real valued function δ on $[a, b]$ such that if $\Pi=\{(x_i, (t_{i-1}, t_i])\}_{i=1}^n$ is any δ-fine P-partition of $[a, b]$ then*

$$\sum_{i=1}^n \left|U(x_i, (t_{i-1}, t_i]) - P\int_{t_{i-1}}^{t_i} U\right| < \varepsilon.$$

2. **Additivity.** E. J. McShane [**1**] proved that if U is P-integrable on $(a, b]$ and $c\in(a, b)$ then U is P-integrable on $(a, c]$ and $(c, b]$ and that

Received by the editors June 15, 1972.
AMS (MOS) subject classifications (1970). Primary 26A42; Secondary 28A25.
Key words and phrases. Integration, indefinite integration, additivity.
[1] This work was partially supported by Arkansas State University research funds.

$P\int_a^b U = P\int_a^c U + P\int_c^b U$. The following theorem gives necessary and sufficient conditions for a form of the converse to hold.

THEOREM 2.1. *Suppose $c \in (a, b)$ and U is P-integrable on $(a, c]$ and $(c, b]$. Then each of the following statements is equivalent to each of the others*:

(i) *U is integrable on $(a, b]$,*

(ii) *if $\varepsilon > 0$, there is a positive number γ such that if* $\sup\{a, c-\gamma\} < x < c < y < \inf\{b, c+\gamma\}$, *then* $|U(c, (x, c]) + U(c, (c, y]) - U(c, (x, y])| < \varepsilon$, *and*

(iii) *if $\varepsilon > 0$ there is a positive number λ such that if*

$$\sup\{a, c - \lambda\} < x < c < y < \inf\{b, c + \lambda\},$$

then $|P\int_x^c U + P\int_c^y U - U(c, (x, y])| < \varepsilon$.

PROOF. Suppose (i) is true and $\varepsilon > 0$. By Lemma K there is a positive function δ on $[a, b]$ satisfying Lemma K for $\varepsilon/3$. Then if $\sup\{a, c-\delta(c)\} < x < c < y < \inf\{c+\delta(c), b\}$,

$$\left|P\int_x^c U - U(c, (x, c])\right| < \varepsilon/3, \tag{1}$$

$$\left|P\int_c^y U - U(c, (c, y])\right| < \varepsilon/3, \quad \text{and} \tag{2}$$

$$\left|P\int_x^y U - U(c, (x, y])\right| < \varepsilon/3. \tag{3}$$

Combining these inequalities yields

$$|U(c, (x, c]) + U(c, (c, y]) - U(c, (x, y])| < \varepsilon.$$

Thus (i)$\Rightarrow$(ii). Also, (3) yields

$$\left|P\int_x^c U + P\int_c^y U - U(c, (x, y])\right| < \varepsilon/3 < \varepsilon.$$

Thus (i)$\Rightarrow$(iii).

Assume (ii) is true, let $\varepsilon > 0$, and let γ be the positive number from (ii) for $\varepsilon/3$. There exist positive functions δ_1 and δ_2 corresponding to the intervals $(a, c]$ and $(c, b]$, which satisfy Lemma K for $\varepsilon/3$. Let

$$\begin{aligned} \delta(x) &= \inf\{\delta_1(x), c - x\} \quad \text{if } x \in [a, c), \\ &= \inf\{\delta_1(c), \delta_2(c), \gamma\} \quad \text{if } x = c, \quad \text{and} \\ &= \inf\{\delta_2(x), x-c\} \quad \text{if } x \in (c, b]. \end{aligned}$$

Suppose $\Pi=\{(t_i, (x_{i-1}, x_i])\}_{i=1}^n$ is a δ-fine P-partition of $(a, b]$. If $c=x_j$ for some j,

$$\begin{aligned}\left|\sum_{i=1}^n U(t_i, (x_{i-1}, x_i]) - P\int_a^c U - P\int_c^b U\right| &\\ \leqq \left|\sum_{i=1}^j U(t_i(x_{i-1}, x_i]) - P\int_a^c U\right| &+ \left|\sum_{i=1}^n U(t_i, (x_{i-1}, x_i]) - P\int_c^b U\right| \\ < \varepsilon/3 + \varepsilon/3 < \varepsilon.&\end{aligned}$$

If $c \in (x_{j-1}, x_j)$ for some j,

$$\begin{aligned}\left|\sum_{i=1}^n U(t_i, (x_{i-1}, x_i]) - P\int_a^c U - P\int_c^b U\right| &\\ \leqq \sum_{i=1}^{j-1}\left|U(t_i, (x_{i-1}, x_i]) - P\int_{x_{i-1}}^{x_i} U\right| &+ \left|U(c, (x_{j-1}, c]) - P\int_{x_{j-1}}^c U\right| \\ + \left|U(c, (c, x_j]) - P\int_c^{x_j} U\right| &+ \sum_{i=j+1}^n \left|U(t_i, (x_{i-1}, x_i]) - P\int_{x_{i-1}}^{x_i} U\right| \\ + |U(c, (x_{j-1}, c]) + U(c, (c, x_j]) - U(c, (x_{j-1}, x_j])| &\\ < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.&\end{aligned}$$

Thus (ii)$\Rightarrow$(i).

Suppose (iii) is true and $\varepsilon>0$. Let λ be the positive number from (iii) for $\varepsilon/3$. Let δ_1 and δ_2 be positive functions on $(a, c]$ and $(c, b]$ respectively which satisfy Lemma K for $\varepsilon/3$. Choose $\gamma=\inf\{\delta_1(c), \delta_2(c), \lambda\}$. Then if $\sup\{a, c-\gamma\}<x<c<y<\inf\{b, c+\gamma\}$,

$$\begin{aligned}|U(c, (x, c]) + U(c, (c, y]) - U(c, (x, y])| &\\ \leqq \left|U(c, (x, c]) - P\int_x^c U\right| &+ \left|U(c, (c, y]) - P\int_c^y U\right| \\ + \left|P\int_x^c U + P\int_c^y U - U(c, (x, y])\right| &\\ < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.&\end{aligned}$$

Thus (iii)$\Rightarrow$(ii).

3. **Indefinite integration.** Suppose U is P-integrable on $(a, b]$ and $g(x)=P\int_a^x U$ for each $x \in [a, b]$. g is called the indefinite integral of U on $[a, b]$.

THEOREM 3.1. *g is bounded on $[a, b]$ if and only if there is a positive function δ on $[a, b]$ and a number M such that if $(c, d]\subset(a, b]$, $t \in [a, b]$, and $(c, d]\subset(t-\delta(t), t+\delta(t))$, then $|U(t, (c, d])|<M$.*

PROOF OF NECESSITY. Suppose g is bounded and let L denote a bound of $|g|$. There is a positive function δ satisfying Lemma K for $\varepsilon=1$. Suppose $(c, d]\subset(a, b]$, $t\in[a, b]$ and $(c, d]\subset(t-\delta(t), t+\delta(t))$. Then there is a δ-fine P-partition of $(a, b]$ with $(t, (c, d])$ as a member. Thus $|U(t, (c, d])-P\int_c^d U|<1$ and $|U(t, (c, d])|<1+|P\int_c^d U|\leqq 1+|g(d)-g(c)|\leqq 1+2L$.

PROOF OF SUFFICIENCY. Suppose U, δ, and M satisfy the hypothesis and assume g is not bounded. Let $\{t_i\}$ denote an increasing sequence of numbers in $[a, b]$ such that $|g(t_i)|>i$ for each i and let $t\in(a, b]$ denote the limit of $\{t_i\}$. If no such sequence exists, there would be a similar decreasing sequence and a similar argument would follow. By Lemma K, there exists a positive function δ' on $[a, t]$ such that if

$$\Pi = \{(t_i, (x_{i-1}, x_i])\}_{i=1}^n$$

is a δ'-fine P-partition of $(a, t]$, then

$$\sum_{i=1}^n \left|U(t_i, (x_{i-1}, x_i]) - P\int_{x_{i-1}}^{x_i} U\right| < 1.$$

Let $\gamma=\inf\{\delta(t), \delta'(t)\}$ and j a positive integer such that $0<|t_j-t|<\gamma$ and $|g(t_i)-g(t)|>M+1$. Since there is a δ'-fine P-partition of $(a, t]$ with $(t, (t_j, t])$ an element,

$$\left|U(t, (t_j, t]) - \int_{t_j}^t U\right| < 1,$$

thus $|g(t)-g(t_j)|-1<|U(t, (t_j, t])|$ and $M<|U(t_j, (t_j, t])|$ which contradicts the hypothesis.

THEOREM 3.2. *g is continuous on $[a, b]$ if and only if for each $\varepsilon>0$ there is a positive function δ on $[a, b]$ such that if $(c, d]\subset(a, b]$, $t\in[a, b]$ and $(c, d]\subset(t-\delta(t), t+\delta(t))$, then $|U(t, (c, d])|<\varepsilon$.*

PROOF OF NECESSITY. Suppose g is on $[a, b]$ and $\varepsilon>0$. Let δ' be positive real from the definition of uniform continuity for $\varepsilon/2$ and δ'' be the positive function from Lemma K for $\varepsilon/2$. Let $\delta(t)=\inf\{\delta', \delta''(t)\}$. Then if $t\in[a, b]$ and $(p, q]\subset(a, b]$ such that $(p, q]\subset(t-\delta(t), t+\delta(t))$, by Lemma K

$$\left|U(t, (p, q]) - P\int_p^q U\right| < \varepsilon/2,$$

$$|U(t, (p, q])| < \varepsilon/2 + |g(q) - g(p)|,$$

and since $|p-q|<\delta$, $|U(t, (p, q])|<\varepsilon$.

PROOF OF SUFFICIENCY. Suppose U satisfies the hypothesis and $\varepsilon>0$.

Let δ be the positive function from the hypothesis for $\varepsilon/2$ and δ'' be the positive function from Lemma K for $\varepsilon/2$. Let $\delta=\inf\{\delta', \delta''\}$. Then if $x, y \in [a, b]$ and $(x, y] \subset (x-\delta(x), x+\delta(x))$,

$$\begin{aligned} |g(x) - g(y)| &= \left| P\int_x^y U \right| \\ &\leqq \left| P\int_x^y U - U(x, (x, y]) \right| + |U(x, (x, y])| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus g is continuous.

THEOREM 3.3. *g is of bounded variation on $[a, b]$ if and only if $|U|$ is P-integrable on $(a, b]$. Furthermore, if g is of bounded variation on $[a, b]$, then the total variation of g on $[a, b]$ is $\int_a^b |U|$.*

PROOF OF NECESSITY. Suppose g is of bounded variation and let Vg denote the total variation of g on $[a, b]$ and let $\varepsilon>0$. There is a partition $a=x_0<x_1<\cdots<x_n=b$ such that $0 \leqq Vg-\sum_{i=1}^n |g(x_i)-g(x_{i-1})|<\varepsilon/2$.

Let

$$\begin{aligned} \delta'(t) &= \inf\{x_i - t, t - x_{i-1}\} \quad \text{if } t \in (x_{i-1}, x_i), \\ &= \tfrac{1}{2} \inf\{x_i - x_{i-1}, x_{i+1} - x_i\} \quad \text{if } t = x_i \text{ for } i = 1, \cdots, n-1, \\ &= \tfrac{1}{2}(x_n - x_{n-1}) \quad \text{if } t = x_n, \\ &= \tfrac{1}{2}(x_1 - x_0) \quad \text{if } t = x_0. \end{aligned}$$

Let δ'' be the positive function from Lemma K for $\varepsilon/2$ and $\delta=\inf\{\delta', \delta''\}$.

Suppose $\Pi=\{(t_i, (y_{i-1}, y_i])\}_{i=1}^m$ is a δ-fine P-partition of $(a, b]$. Then $a=y_2<y_1<\cdots<y_m=b$ is a refinement of $a=x_0<x_1<\cdots<x_m=b$ and thus $0 \leqq Vg-\sum_{i=1}^m |g(y_i)-g(y_{i-1})|<\varepsilon/2$ and $0 \leqq Vg-\sum_{i=1}^m |P\int_{y_{i-1}}^{y_i} U|<\varepsilon/2$. It follows from Lemma K that

$$\left| \sum_{i=1}^m |U(t_i, (y_{i-1}, y_i])| - \sum_{i=1}^m \left| P\int_{y_{i-1}}^{y_i} U \right| \right| < \frac{\varepsilon}{2}.$$

Combining inequalities yields

$$\left| \sum_{i=1}^m |U(t_i, (y_{i-1}, y_i])| - Vg \right| < \varepsilon.$$

Therefore $|U|$ is P-integrable and $P\int_a^b |U|=Vg$.

PROOF OF SUFFICIENCY. Suppose $|U|$ is P-integrable and $\varepsilon>0$. Let δ'' be a positive function which satisfies Lemma K for both U and $|U|$ for $\varepsilon/2$.

Let $a=x_0<x_1<\cdots<x_n=b$ be any partition of $[a, b]$. Define δ as in the necessary proof and let $\Pi=\{(t_i, (y_{i-1}, y_i])\}_{i=1}^m$ be any δ-fine P-partition of $(a, b]$.

Lemma K yields

$$\sum_{i=1}^m \left| U(t_i, (y_{i-1}, y_i]) - P\int_{y_{i-1}}^{y_i} U \right| < \frac{\varepsilon}{2}$$

and thus

$$\sum_{i=1}^m \left| \, |U(t_i, (y_{i-1}, y_i])| - |g(y_i) - g(y_{i-1})| \, \right| < \frac{\varepsilon}{2}.$$

Also, Lemma K gives

$$\sum_{i=1}^m \left| \, |U(t_i, (y_{i-1}, y_i]| - P\int_{y_{i-1}}^{y_i} |U| \, \right| < \frac{\varepsilon}{2}.$$

Combining inequalities yields

$$\left| \sum_{i=1}^m |g(y_i) - g(y_{i-1})| - P\int_a^b |U| \right| < \varepsilon.$$

Therefore it follows from the definition of bounded variation that g is of bounded variation and that Vg is $P\int_a^b |U|$.

References

1. E. J. McShane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. Amer. Math. Soc. No. 88 (1969). MR **42** #436.

2. A. Kolmogoroff, *Unterschungen über den Integralbegriff*, Math. Ann. **103** (1930), 654–696.

3. Ralph Henstock, *Theory of integration*, Butterworths, London, 1963. MR **28** #1274.

Department of Mathematics, Arkansas State University, State University, Arkansas 72467