

## REFLEXIVITY OF $L(E, F)$

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ABSTRACT. Let  $E$  and  $F$  be Banach spaces and denote by  $L(E, F)$  (resp.,  $K(E, F)$ ) the space of all bounded linear operators (resp., all compact operators) from  $E$  to  $F$ . In this note the following theorem is proved: *If  $E$  and  $F$  are reflexive and one of  $E$  and  $F$  has the approximation property then the following are equivalent:*

- (i)  $L(E, F)$  is reflexive,
- (ii)  $L(E, F) = K(E, F)$ ,
- (iii) if  $T \neq 0 \in L(E, F)$ , then  $\|T\| = \|Tx\|$  for some  $x \in E$ ,  $\|x\| = 1$ .

This result extends a recent result of Ruckle (Proc. Amer. Math. Soc. **34** (1972), 171–174) who showed (i) and (ii) are equivalent when both  $E$  and  $F$  have the approximation property. Moreover the proof suggests strongly that the assumption of the approximation property may be dropped.

The purpose of this note is to call attention to an unsolved problem in Banach space theory whose complete solution seems to be quite elusive and to make a contribution toward the complete solution by opening a new avenue of approach.

Let  $E$  and  $F$  be Banach spaces and denote by  $L(E, F)$  (resp.,  $K(E, F)$ ) the space of all bounded linear operators (resp., all compact operators) from  $E$  to  $F$ . A problem which is as yet unsolved is: *Characterize those spaces  $L(E, F)$  which are reflexive.* A partial solution has been given in [4] and [7], namely:

**THEOREM 1.** *If  $E$  and  $F$  are reflexive and both  $E$  and  $F$  have the approximation property then  $L(E, F)$  is reflexive if and only if  $L(E, F) = K(E, F)$ .*

Our purpose here is to prove a result (Theorem 2) which is both an extension of, and an improvement on, Theorem 1. In particular, we give another characterization of those spaces  $L(E, F)$  which are reflexive and at the same time show that Theorem 1 is valid under the weaker assumption that either  $E$  or  $F$  has the approximation property. Also, our proof avoids

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the use of the deep theorem of Grothendieck [2] upon which the proof of Theorem 1 given in [4] and [7] is based. More importantly, however, our proof suggests strongly a possible way to avoid the use of the approximation property altogether.

**THEOREM 2.** *Let  $E$  and  $F$  be Banach spaces for which either  $E$  or  $F$  has the approximation property. Then the following are equivalent:*

- (i)  $L(E, F)$  is reflexive,
- (ii)  $L(E, F) = K(E, F)$ ,
- (iii) if  $T \neq 0 \in L(E, F)$  then  $T$  has a norming point—i.e. there exists  $x \in E$ ,  $\|x\| = 1$ , such that  $\|T\| = \|Tx\|$ .

**PROOF.** (i)  $\Rightarrow$  (ii). Suppose  $T \in L(E, F)$  is not compact. Then (cf. e.g. Rosenthal [6]) there is a sequence  $(x_i) \subset E$  for which  $(x_i)$  converges weakly to zero but  $\inf_i \|Tx_i\| > 0$ . Since  $(Tx_i)$  also converges weakly to zero we can find a subsequence  $(x_{n_i})$  of  $(x_i)$  such that both  $(x_{n_i})$  and  $(Tx_{n_i})$  are basic sequences in  $E$  and  $F$ , respectively [1]. Let  $(g_i) \subset F^*$  be biorthogonal to  $(Tx_i)$ . Recall that if  $\gamma$  denotes the greatest crossnorm then  $L(E, F) = (E \otimes_\gamma F^*)^*$  [8] and  $(x_{n_i} \otimes g_i) \subset E \otimes_\gamma F^*$ . If  $L(E, F)$  is reflexive then so is  $E \otimes_\gamma F^*$  and consequently a subsequence of  $(x_{n_i} \otimes g_i)$  (which we assume to be  $(x_{n_i} \otimes g_i)$  for notational convenience) is weakly convergent, say to  $z \in E \otimes_\gamma F^*$ . If  $\lambda$  denotes the "least" crossnorm [8] then since  $\gamma \geq \lambda$  it must be that  $(x_{n_i} \otimes g_i)$  also converges weakly to  $z$  in  $E \otimes_\lambda F^*$  (where we identify  $x_{n_i} \otimes g_i$  and  $z$  with their images in  $E \otimes_\lambda F^*$  under the injection  $E \otimes_\gamma F^* \rightarrow E \otimes_\lambda F^*$ ). But  $(x_{n_i} \otimes g_i)$  is a basic sequence in  $E \otimes_\lambda F^*$  [3] and so  $z = 0$  as an element of  $E \otimes_\lambda F^*$ . Since by assumption either  $E$  or  $F$  has the approximation property it follows that  $z = 0$  in  $E \otimes_\gamma F^*$  also [2], a contradiction to the fact that  $T \in (E \otimes_\gamma F^*)^*$  and  $\langle T, x_{n_i} \otimes g_i \rangle = 1$  for all  $i$ . Therefore  $L(E, F)$  cannot be reflexive, and (i)  $\Rightarrow$  (ii).

(ii)  $\Rightarrow$  (iii). Suppose  $L(E, F) = K(E, F)$  and let  $T \neq 0$  be an element of  $L(E, F)$ . By definition of the norm of  $T$  there is a sequence  $(x_i)$  in  $E$  for which  $\|x_i\| = 1$  for all  $i$  and  $(\|Tx_i\|)$  converges to  $\|T\|$ . Since  $E$  is reflexive, a subsequence  $(x_{n_i})$  is weakly convergent to some  $x \in E$  with  $\|x\| \leq 1$  and  $(Tx_{n_i})$  converges in norm to  $Tx$  [6]. It now follows easily that  $\|Tx\| = \|T\|$  so (ii)  $\Rightarrow$  (iii).

(iii)  $\Rightarrow$  (i). Recall that  $L(E, F) = (E \otimes_\gamma F^*)^*$  so if  $T \in L(E, F)$  has a norming point  $x$  then there is a  $g \in F^*$  with  $\|g\| = 1$  for which  $\|T\| = \|Tx\| = \langle Tx, g \rangle$ , implying  $\langle T, x \otimes g \rangle = \|T\|$  for  $\|x \otimes g\| = 1$  and  $x \otimes g \in E \otimes_\gamma F^*$ . That is, every element of  $(E \otimes_\gamma F^*)^*$  attains its norm on the unit ball in  $E \otimes_\gamma F^*$  so  $E \otimes_\gamma F^*$  is reflexive [5]. It follows, of course, that  $L(E, F)$  is reflexive, and (iii)  $\Rightarrow$  (i).

The assumption that  $E$  or  $F$  have the approximation property is used only in the proof of (i)  $\Rightarrow$  (ii) and an inspection of this proof shows that

the assumption could be dispensed with if the answer to the following question was in the affirmative.

*Question.* If  $(x_i)$  is a basic sequence in  $E$  and  $(y_i)$  is a sequence in  $F$  for which  $0 < \inf_i \|y_i\| \leq \sup_i \|y_i\| < +\infty$ , is  $(x_i \otimes y_i)$  a basic sequence in  $E \otimes_\lambda F$ ?

This question was first posed in [3] and seems to be difficult. However the fact that it is true for  $E \otimes_\lambda F$  (a fact used in (i)  $\Rightarrow$  (ii)) and it is true if  $(x_i)$  is a basis for  $E$  leads one to conjecture that the answer to the question is "yes".

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