PRODUCT INTEGRALS AND EXPONENTIALS IN COMMUTATIVE BANACH ALGEBRAS

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Abstract. Functions are from \( \mathbb{R} \times \mathbb{R} \) to \( X \), where \( \mathbb{R} \) represents the real numbers and \( X \) represents a commutative Banach algebra with identity element. The function \( G \in OC^\alpha \) on \( [a, b] \) only if \( \alpha \int_a^b (1+G) \) exists and is not zero and there exists a subdivision \( D \) of \( [a, b] \) and a number \( B \) such that if \( J \) is a refinement of \( D \), then \( \prod J (1+G)^{-1} \) exists and \( ||\prod J (1+G)^{-1}|| < B \). If \( |G| < 1 \) on \( [a, b] \), then each of the following consists of two equivalent statements: A. (1) \( G \in OC^\alpha \) on \( [a, b] \), and (2) \( \int_a^b \ln(1+G) \) exists. B. (1) \( G \in OC^\alpha \) on \( [a, b] \) and \( \int_a^b |1+G-\prod (1+G)| = 0 \), and (2) \( \int_a^b \ln(1+G)-\int \ln(1+G)| = 0 \). Further, if \( \beta > 0 \), \( |G| < 1 - \beta \) on \( [a, b] \), each of \( G(p, p^+), G(p^+, p) \), \( G(p^+, p^+) \) and \( G(p^-, p^-) \) exist for \( p \in [a, b] \), \( \int_a^b |G^2 - \prod G^2| = 0 \) and \( G^2 \) has bounded variation on \( [a, b] \), then each of the following consists of two equivalent statements: C. (1) \( G \in OC^\alpha \) on \( [a, b] \), and (2) \( \int_a^b G \) exists. D. (1) \( G \in OC^\alpha \) on \( [a, b] \) and \( \int_a^b |1+G-\prod (1+G)| = 0 \), and (2) \( \int_a^b G - \prod G = 0 \).

In two recent papers relationships between product integrals and exponentials are investigated for real valued functions. W. P. Davis and J. A. Chatfield [1, Theorem 3, p. 744] show that, if \( \int_a^b G^2 = 0 \), then \( \int_a^b G \) exists if and only if \( \alpha \int_a^b (1+G) \) exists and is not zero. Furthermore,

\[
a \int_a^b (1 + G) = \exp \int_a^b G.
\]

The author [4, Theorem 5] shows that, if \( \beta > 0 \), \( |G| < 1 - \beta \) on \( [a, b] \) and \( \int_a^b G^2 \) exists, then \( \int_a^b G \) exists if and only if \( \alpha \int_a^b (1+G) \) exists and is not zero. Furthermore,

\[
a \int_a^b (1 + G) = \exp \int_a^b \ln(1 + G).
\]

We extend these results to functions with values in a commutative Banach algebra with identity element.
Let $X$ be a Banach algebra with identity element. Thus,

1. $X$ is a complete normed linear space with real or complex scalars,
2. $X$ is a linear associative algebra with unit 1, and
3. If $x, y \in X$, then $|xy| \leq |x| |y|$ and $|1| = 1$.

Further, if $x \in X$ and $|x - 1| < 1$, then

1. $x^{-1}$ exists and is $\sum_{n=0}^{\infty} (1 - x)^n$, and
2. $|x^{-1}| \leq [1 - |1 - x|]^{-1}$.

Exponential and logarithmic functions are defined by the equations

1. $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ for $x \in X$, and
2. $\ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1}(x-1)^n/n$ for $x \in X$ and $|x - 1| < 1$.

These functions are continuous in their domains of definition, and

1. $\exp[\ln(x)] = x$ if $|x - 1| < 1$, and
2. $\ln[\exp(x)] = x$ if $|x - 1| < \ln 2$.

In addition, if $xy = yx$, then

1. $[\exp(x)][\exp(y)] = \exp(x + y)$, and
2. $\ln(xy) = \ln(x) + \ln(y)$.

If $|x - 1| < 1$, $|y - 1| < 1$, $xy = yx$ and $z = xy$, then (2) is used to define $\ln(z)$.

Consult P. R. Masani [7, pp. 151–152] for additional details and background. Note that $X$ is not the same as the ring $N$ considered by J. S. MacNerney [6, p. 150] and B. W. Helton [2, pp. 298–299].

All integrals and definitions are of the subdivision-refinement type, $R$ denotes the set of all real numbers, and functions are $X$-valued and understood to be defined only on $\{x, y\} \in R \times R$ such that $x < y$. The statements that $G$ is bounded, $G \in OB^o$, $G \in OP^o$ and $G \in OU^o$ on $[a, b]$ mean there exists a subdivision $D$ of $[a, b]$ and a number $B$ such that if $J = \{x_q\}_0^n$ is a refinement of $D$, then

1. $|G(u)| < B$ for $u \in J(I)$,
2. $\sum_{J(I)} |G| < B$,
3. $\prod_{1 \leq i \leq j \leq n} |1 + G_{ij}| < B$ for $1 \leq i \leq j \leq n$, and
4. $\prod_{1 \leq i \leq j \leq n} (1 + G_{ij})^{-1}$ exists and $\prod_{1 \leq i \leq j \leq n} (1 + G_{ij})^{-1} < B$ for $1 \leq i \leq j \leq n$, respectively, where $G_{ij} = G(x_{ij-1}, x_{ij})$ and $J(I) = \{x_{ij-1}, x_{ij}\}$.

Similarly, statements of the form $G > b$ should be interpreted in terms of subdivisions and refinements. Further, $G \in OC^o$ on $[a, b]$ only if

1. $\prod_{1 \leq i \leq n} (1 + G)$ exists and is not zero, and
2. there exists a subdivision $D$ of $[a, b]$ and a number $B$ such that if $J$ is a refinement of $D$, then $\prod_{J(I)} (1 + G)$ exists and $|\prod_{J(I)} (1 + G)| < B$.

Also,

1. $G \in OA^o$ on $[a, b]$ only if $\int_a^b G \text{ exists and } \int_a^b |G - \{G\}| = 0$,
2. $G \in OM^o$ on $[a, b]$ only if $\exists \prod_{1 \leq i \leq n} (1 + G \text{ exists for } a \leq x < y \leq b$ and $\int_a^b |1 + G - \{1 + G\}| = 0$, and
3. $G \in OL^o$ on $[a, b]$ only if $\lim_{x \to x^+} G(p, x)$, $\lim_{x \to x^-} G(x, p)$, $\lim_{x \to y^+} G(x, y)$ and $\lim_{x \to y^-} G(x, y)$ exist for $p \in [a, b]$. 

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The statement that the sequence of functions \( \{S_n\}_n \) converges uniformly to a function \( S \) on \([a, b]\) means if \( \varepsilon > 0 \) then there exists a subdivision \( D \) of \([a, b]\) and a positive integer \( N \) such that if \( J \) is a refinement of \( D \), \( u \in J(I) \) and \( n > N \), then \( |S_n(u) - S(u)| < \varepsilon \). See B. W. Helton [2] for more details.

**Theorem 1.** Suppose \( X \) is commutative. If \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( X \) such that \( |G| < 1 \) on \([a, b]\), then the following statements are equivalent:

1. \( G \in OC^\circ \) on \([a, b]\), and
2. \( \int_a^b \ln(1+G) \) exists.

Furthermore, \( \int_a^b (1+G) = \exp \int_a^b \ln(1+G) \).

**Lemma 1.** If \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( X \) such that \( |G| < 1 \) on \([a, b]\) and \( G \in OC^\circ \) on \([a, b]\), then \( G \in OP^\circ \) and \( OU^\circ \) on \([a, b]\).

**Theorem 2.** Suppose \( X \) is commutative. If \( G \) is a function from \( \mathbb{R} \times \mathbb{R} \) to \( X \) such that \( |G| < 1 \) on \([a, b]\), then the following statements are equivalent:

1. \( G \in OC^\circ \) and \( OM^\circ \) on \([a, b]\), and
2. \( \ln(1+G) \in O\mathcal{A}^\circ \) on \([a, b]\).

Furthermore, \( \int_a^b (1+G) = \exp \int_a^b \ln(1+G) \).

**Proof.** (1)→(2) It follows from Theorem 1 that \( \int_a^b \ln(1+G) \) exists. Hence, it is only necessary to show that

\[
\int_a^b \left| \ln(1+G) - \int \ln(1+G) \right| = 0.
\]

Note that \( G \in OP^\circ \) and \( OU^\circ \) on \([a, b]\) [Lemma 1]. Let \( \varepsilon > 0 \). There exists a subdivision \( D \) of \([a, b]\) and a number \( B \) such that if \( J = \{x_q\}_n \) is a refinement of \( D \) and \( J_q \) is a subdivision of \([x_{q-1}, x_q]\) for \( 1 \leq q \leq n \), then

1. \(|\prod_j (1+G_q)^{-1}| < B \) for \( 1 \leq q \leq n \),
2. \(|1 + \sum_{q=1}^n (-1)^{i-1} \{[1+G_q][\prod_{j \neq q} (1+G_j)]^{-1} - 1\}^{i-1}/i| < B \) for \( 1 \leq q \leq n \),
3. \(|[1+G_q][\prod_{j \neq q} (1+G_j)]^{-1} - 1| < 1 \) for \( 1 \leq q \leq n \), and
4. \( \sum_{q=1}^n |1+G_q - \prod_{j \neq q} (1+G_j)| < \varepsilon/2B^2 \).

Let \( J = \{x_q\}_n \) be a refinement of \( D \). For \( 1 \leq q \leq n \), let \( J_q \) be a subdivision of \([x_{q-1}, x_q]\) such that

\[
\left| \sum_{J_q} \ln(1+G) - \int_{x_{q-1}}^{x_q} \ln(1+G) \right| < \varepsilon/2n.
\]
Thus,
\[
\sum_{q=1}^{n} \left| \ln(1 + G_q) - \int_{x_{q-1}}^{x_q} \ln(1 + G) \right|
\leq \sum_{q=1}^{n} \left| \ln \left( [1 + G_q] \left[ \prod_{J_q(I)} (1 + G) \right]^{-1} \left[ \prod_{J_q(I)} (1 + G) \right] \right) - \ln(1 + G) \right|
\]
\[
+ \sum_{q=1}^{n} \sum_{J_q(I)} \ln(1 + G) - \int_{x_{q-1}}^{x_q} \ln(1 + G)
\]
\[
< \sum_{q=1}^{n} \left| \ln \left( [1 + G_q] \left[ \prod_{J_q(I)} (1 + G) \right]^{-1} \right) \right| + n(e/2n)
\]
\[
= \sum_{q=1}^{n} \left| \sum_{i=1}^{\infty} (-1)^{i-1} \left( [1 + G_q] \left[ \prod_{J_q(I)} (1 + G) \right]^{-1} - 1 \right)^i/i \right| + \epsilon/2
\]
\[
\leq B \sum_{q=1}^{n} \left| [1 + G_q] \left[ \prod_{J_q(I)} (1 + G) \right]^{-1} - 1 \right| + \epsilon/2
\]
\[
\leq B \sum_{q=1}^{n} \left| 1 + G_q - \prod_{J_q(I)} (1 + G) \right| + \epsilon/2
\]
\[
\leq B^2 \sum_{q=1}^{n} \left| 1 + G_q - \prod_{J_q(I)} (1 + G) \right| + \epsilon/2
\]
\[
< B^2(e/2B) + \epsilon/2 = \epsilon.
\]
Therefore, \( \ln(1 + G) \in O A^o \) on \( [a, b] \).

**Proof.** (2)→(1) It follows from Theorem 1 that
(1) \( G \in O C^o \) on \( [a, b] \),
(2) \( \prod_{J_q(I)} (1 + G) \) exists for \( a \leq x < y \leq b \), and \( a \prod_{J_q(I)} (1 + G) = \exp \int_a^x \ln (1 + G) \).

Hence, it is only necessary to show that
\[
\int_a^b \left| 1 + G - \prod_{J_q(I)} (1 + G) \right| = 0.
\]

Let \( \epsilon > 0 \). Lemma 1 implies that \( G \in O P^o \) and \( O U^o \) on \( [a, b] \). There exists a subdivision \( D \) of \( [a, b] \) and a number \( B \) such that if \( J = \{x_{q}\} \) is a subdivision of \( D \) and \( J_q \) is a subdivision of \( [x_{q-1}, x_q] \) for \( 1 \leq q \leq n \), then
(1) \( \prod_{J_q(I)} (1 + G_q) < B \) for \( 1 \leq r \leq s \leq n \),
(2) \( \left| [1 + G_q] \prod_{J_q(I)} (1 + G) \right| - 1 < 1 \) for \( 1 \leq q \leq n \),

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(3) $|\{P_q\}^{-1}| < B$ for $1 \leq q \leq n$, where

$$P_q = 1 + \sum_{i=0}^{\infty} ( -1)^{i-1}\left\{1 + G_q\right\}\left[\prod_{J_q(i)} (1 + G)\right]^{-1} - 1 \right\}^{-1}/i,$$

and

$$\sum_{q=1}^{n} |\ln(1+G_q) - \sum_{J_q(i)} \ln(1+G)| < \epsilon/2B^2.$$

Let $J = \{x_q\}_{q=1}^{n}$ be a refinement of $D$. Further, for $1 \leq q \leq n$, let $J_q$ be a subdivision of $[x_{q-1}, x_q]$ such that

$$\left|\prod_{J_q(i)} (1 + G) - \prod_{x_{q-1}}^{x_q} (1 + G)\right| < \epsilon/2n.$$

Thus,

$$\sum_{q=1}^{n} \left|1 + G_q - \prod_{x_{q-1}}^{x_q} (1 + G)\right| < \sum_{q=1}^{n} \left|1 + G_q - \prod_{J_q(i)} (1 + G)\right| + n(\epsilon/2n)

\leq \sum_{q=1}^{n} \left|[1 + G_q]\left[\prod_{J_q(i)} (1 + G)\right]^{-1} - 1\right| \left|\prod_{J_q(i)} (1 + G)\right| + \epsilon/2

\leq B \sum_{q=1}^{n} \left|[1 + G_q]\left[\prod_{J_q(i)} (1 + G)\right]^{-1} - 1\right| + \epsilon/2

\leq B \sum_{q=1}^{n} \left|[1 + G_q]\left[\prod_{J_q(i)} (1 + G)\right]^{-1} - 1\right| + \epsilon/2

= B^2 \sum_{q=1}^{n} \left|\ln(1 + G_q) - \ln(1 + G)\right| + \epsilon/2

= B^2 \sum_{q=1}^{n} \left|\ln(1 + G_q) - \ln(1 + G)\right| + \epsilon/2

< B^2(\epsilon/2B^2) + \epsilon/2 = \epsilon.$$

Therefore, $G \in O\alpha^0$ on $[a, b]$.

**Lemma 2.** If $G$ is a function from $R \times R$ to $X$, $G \in O\beta^0$ on $[a, b]$, $(S_n)_{n=1}^{\infty}$ is a sequence of functions from $R \times R$ to $X$ converging uniformly to a bounded function $S$ on $[a, b]$ and $S_n G \in O\alpha^0$ on $[a, b]$, then $SG \in O\alpha^0$ on $[a, b]$. If $\lim_{n \to \infty} \int_{a}^{b} S_n G = S \int_{a}^{b} G$, then $SG = \lim_{n \to \infty} \int_{a}^{b} S_n G$.

The proof of Lemma 2 is straightforward and, therefore, we omit it.
Lemma 3. If $H$ and $G$ are functions from $\mathbb{R} \times \mathbb{R}$ to $X$, $H \in O\Lambda^0$ on $[a, b]$ and $G \in O\Lambda^0$ and $O\Lambda^0$ on $[a, b]$, then $HG \in O\Lambda^0$ and $OM^0$ on $[a, b]$ [3, Theorem 2, p. 494].

Lemma 4. If $\beta > 0$ and $G$ is a function from $\mathbb{R} \times \mathbb{R}$ to $X$ such that $|G| < 1 - \beta$ on $[a, b]$, $G \in O\Lambda^0$ on $[a, b]$ and $G^2 \in O\Lambda^0$ and $O\Lambda^0$ on $[a, b]$, then

$$\sum_{n=2}^{\infty} (-1)^{n-1} G^n/n$$

is in $O\Lambda^0$ on $[a, b]$.

Proof. Since $G \in O\Lambda^0$ on $[a, b]$,

$$\sum_{n=2}^{p} (-1)^{n-1} (G^{n-2})/n$$

is in $O\Lambda^0$ on $[a, b]$ for $p=2, 3, 4, \ldots$. Thus,

$$\left[ \sum_{n=2}^{p} (-1)^{n-1} (G^{n-2})/n \right] G^2$$

is in $O\Lambda^0$ on $[a, b]$ for $p=2, 3, 4, \ldots$ [Lemma 3]. Therefore, since

$$\left\{ \sum_{n=2}^{p} (-1)^{n-1} (G^{n-2})/n \right\}_{p=2}^{\infty}$$

converges uniformly to

$$\sum_{n=2}^{\infty} (-1)^{n-1} (G^{n-2})/n$$

on $[a, b]$, it follows from Lemma 2 that

$$\left[ \sum_{n=2}^{\infty} (-1)^{n-1} (G^{n-2})/n \right] G^2 = \sum_{n=2}^{\infty} (-1)^{n-1} G^n/n$$

is in $O\Lambda^0$ on $[a, b]$.

We note that a similar justification of Lemma 4 for real valued functions is contained in the proof of Theorem 5 (1 $\rightarrow$ 2) of a previous paper by the author [4].

Theorem 3. Suppose $X$ is commutative. If $\beta > 0$ and $G$ is a function from $\mathbb{R} \times \mathbb{R}$ to $X$ such that $|G| < 1 - \beta$ on $[a, b]$, $G \in O\Lambda^0$ on $[a, b]$ and $G^2 \in O\Lambda^0$ and $O\Lambda^0$ on $[a, b]$, then the following statements are equivalent:

1. $G \in O\Lambda^0$ on $[a, b]$, and
2. $\int_a^b G$ exists.

Furthermore, $\frac{\alpha}{\beta} \prod (1 + G) = \exp \int_a^b \ln(1 + G)$.
Proof. It follows from Lemma 4 that
\[ \int_a^b \sum_{n=2}^{\infty} (-1)^{n-1} G^n/n \]
exists. Hence, if (1) is true, then \( \int_a^b G \) exists since Theorem 1 implies that
\[ \int_a^b \ln(1 + G) = \int_a^b \sum_{n=1}^{\infty} (-1)^{n-1} G^n/n \]
exists. Also, if (2) is true, then Theorem 1 implies that (1) is true since
\[ \int_a^b \sum_{n=1}^{\infty} (-1)^{n-1} G^n/n = \int_a^b \ln(1 + G) \]
exists. In addition, Theorem 1 also implies that
\[ a \prod_b^b (1 + G) = \exp \int_a^b \ln(1 + G). \]

Theorem 4. Suppose \( X \) is commutative. If \( \beta > 0 \) and \( G \) is a function from \( R \times R \) to \( X \) such that \( |G| < 1 - \beta \) on \( [a, b] \), \( G \in OL^\circ \) on \( [a, b] \) and \( G^2 \in OA^\circ \) and \( OB^\circ \) on \( [a, b] \), then the following statements are equivalent:

(1) \( G \in OC^\circ \) and \( OM^\circ \) on \( [a, b] \), and
(2) \( G \in OA^\circ \) on \( [a, b] \).

Furthermore, \( a \prod_b^b (1 + G) = \exp \int_a^b \ln(1 + G). \)

Proof. It follows from Lemma 4 that
\[ \sum_{n=2}^{\infty} (-1)^{n-1} G^n/n \]
is in \( OA^\circ \) on \( [a, b] \). Hence, if (1) is true, then \( G \in OA^\circ \) on \( [a, b] \) since Theorem 2 implies that
\[ \ln(1 + G) = \sum_{n=2}^{\infty} (-1)^{n-1} G^n/n \]
is in \( OA^\circ \) on \( [a, b] \). Also, if (2) is true, then Theorem 2 implies that (1) is true since
\[ \sum_{n=1}^{\infty} (-1)^{n-1} G^n/n = \ln(1 + G) \]
is in \( OA^\circ \) on \( [a, b] \). In addition, Theorem 2 also implies that
\[ a \prod_b^b (1 + G) = \exp \int_a^b \ln(1 + G). \]
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