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PRODUCT INTEGRALS AND EXPONENTIALS IN COMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Functions are from $R\times R$ to X, where R represents the real numbers and X represents a commutative Banach algebra with identity element. The function $G\in OC^\circ$ on $[a,b]$ only if ${}_a\prod^b(1+G)$ exists and is not zero and there exists a subdivision D of $[a,b]$ and a number B such that if J is a refinement of D, then $[\prod_J(1+G)]^{-1}$ exists and $|[\prod_J(1+G)]^{-1}|<B$. If $|G|<1$ on $[a,b]$, then each of the following consists of two equivalent statements: A. (1) $G\in OC^\circ$ on $[a,b]$, and (2) $\int_a^b \ln(1+G)$ exists. B. (1) $G\in OC^\circ$ on $[a,b]$ and $\int_a^b|1+G-\prod(1+G)|=0$, and (2) $\int_a^b|\ln(1+G)-\int\ln(1+G)|=0$. Further, if $\beta>0$, $|G|<1-\beta$ on $[a,b]$, each of $G(p,p^+)$, $G(p^-,p)$, $G(p^+,p^+)$ and $G(p^-,p^-)$ exist for $p\in[a,b]$, $\int_a^b|G^2-\int G^2|=0$ and G^2 has bounded variation on $[a,b]$, then each of the following consists of two equivalent statements: C. (1) $G\in OC^\circ$ on $[a,b]$, and (2) $\int_a^b G$ exists. D. (1) $G\in 0C^\circ$ on $[a,b]$ and $\int_a^b|1+G-\prod(1+G)|=0$, and (2) $\int_a^b|G-\int G|=0$.

In two recent papers relationships between product integrals and exponentials are investigated for real valued functions. W. P. Davis and J. A. Chatfield [**1**, Theorem 3, p. 744] show that, if $\int_a^b G^2=0$, then $\int_a^b G$ exists if and only if ${}_a\prod^b(1+G)$ exists and is not zero. Furthermore,

$$ {}_a\prod^b(1+G)=\exp\int_a^b G. $$

The author [**4**, Theorem 5] shows that, if $\beta>0$, $|G|<1-\beta$ on $[a,b]$ and $\int_a^b G^2$ exists, then $\int_a^b G$ exists if and only if ${}_a\prod^b(1+G)$ exists and is not zero. Furthermore,

$$ {}_a\prod^b(1+G)=\exp\int_a^b \ln(1+G). $$

We extend these results to functions with values in a commutative Banach algebra with identity element.

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Let X be a Banach algebra with identity element. Thus,

(1) X is a complete normed linear space with real or complex scalars,

(2) X is a linear associative algebra with unit 1, and

(3) if $x, y \in X$, then $|xy| \leqq |x|\,|y|$ and $|1|=1$.

Further, if $x \in X$ and $|x-1|<1$, then

(1) x^{-1} exists and is $\sum_{n=0}^{\infty} (1-x)^n$, and

(2) $|x^{-1}| \leqq [1-|1-x|]^{-1}$.

Exponential and logarithmic functions are defined by the equations

(1) $\exp(x)=\sum_{n=0}^{\infty} x^n/n!$ for $x \in X$, and

(2) $\ln(x)=\sum_{n=1}^{\infty} (-1)^{n-1}(x-1)^n/n$ for $x \in X$ and $|x-1|<1$.

These functions are continuous in their domains of definition, and

(1) $\exp[\ln(x)]=x$ if $|x-1|<1$, and

(2) $\ln[\exp(x)]=x$ if $|x-1|<\ln 2$.

In addition, if $xy=yx$, then

(1) $[\exp(x)][\exp(y)]=\exp(x+y)$, and

(2) $\ln(xy)=\ln(x)+\ln(y)$.

If $|x-1|<1$, $|y-1|<1$, $xy=yx$ and $z=xy$, then (2) is used to define $\ln(z)$. Consult P. R. Masani [**7**, pp. 151–152] for additional details and background. Note that X is not the same as the ring N considered by J. S. MacNerney [**6**, p. 150] and B. W. Helton [**2**, pp. 298–299].

All integrals and definitions are of the subdivision-refinement type, R denotes the set of all real numbers, and functions are X-valued and understood to be defined only on $\{x, y\}$ in $R \times R$ such that $x<y$. The statements that G is bounded, $G \in OB^\circ$, $G \in OP^\circ$ and $G \in OU^\circ$ on $[a, b]$ mean there exists a subdivision D of $[a, b]$ and a number B such that if $J=\{x_q\}_0^n$ is a refinement of D, then

(1) $|G(u)|<B$ for $u \in J(I)$,

(2) $\sum_{J(I)} |G|<B$,

(3) $|\prod_i^j (1+G_q)|<B$ for $1 \leqq i \leqq j \leqq n$, and

(4) $[\prod_i^j (1+G_q)]^{-1}$ exists and $|[\prod_i^j (1+G_q)]^{-1}|<B$ for $1 \leqq i \leqq j \leqq n$, respectively, where $G_q=G(x_{q-1}, x_q)$ and $J(I)=\{[x_{q-1}, x_q]\}_1^n$.

Similarly, statements of the form $G>b$ should be interpreted in terms of subdivisions and refinements. Further, $G \in OC^\circ$ on $[a, b]$ only if

(1) ${}_a\prod^b (1+G)$ exists and is not zero, and

(2) there exists a subdivision D of $[a, b]$ and a number B such that if J is a refinement of D, then $[\prod_{J(I)} (1+G)]^{-1}$ exists and $|[\prod_{J(I)} (1+G)]^{-1}|<B$.

Also,

(1) $G \in OA^\circ$ on $[a, b]$ only if $\int_a^b G$ exists and $\int_a^b |G-\int G|=0$,

(2) $G \in OM^\circ$ on $[a, b]$ only if ${}_x\prod^y (1+G)$ exists for $a \leqq x<y \leqq b$ and $\int_a^b |1+G-\prod (1+G)|=0$, and

(3) $G \in OL^\circ$ on $[a, b]$ only if $\lim_{x \to p^+} G(p, x)$, $\lim_{x \to p^-} G(x, p)$, $\lim_{x,y \to p^+} G(x, y)$ and $\lim_{x,y \to p^-} G(x, y)$ exist for $p \in [a, b]$.

The statement that the sequence of functions $\{S_n\}_1^\infty$ converges uniformly to a function S on $[a, b]$ means if $\varepsilon > 0$ then there exists a subdivision D of $[a, b]$ and a positive integer N such that if J is a refinement of D, $u \in J(I)$ and $n > N$, then $|S_n(u) - S(u)| < \varepsilon$. See B. W. Helton [**2**] for more details.

THEOREM 1. *Suppose X is commutative. If G is a function from $R \times R$ to X such that $|G| < 1$ on $[a, b]$, then the following statements are equivalent*:

(1) $G \in OC^\circ$ *on* $[a, b]$, *and*

(2) $\int_a^b \ln(1+G)$ *exists.*

Furthermore, ${}_a\prod^b (1+G) = \exp \int_a^b \ln(1+G)$.

Theorem 1 follows from the continuity of the exponential and logarithmic functions.

LEMMA 1. *If G is a function from $R \times R$ to X such that $|G| < 1$ on $[a, b]$ and $G \in OC^\circ$ on $[a, b]$, then $G \in OP^\circ$ and OU° on $[a, b]$.*

Lemma 1 follows by the same argument as used in a previous result of the author [**5**, Theorem 2] since $|G| < 1$ implies that G is bounded and that the necessary inverses exist.

THEOREM 2. *Suppose X is commutative. If G is a function from $R \times R$ to X such that $|G| < 1$ on $[a, b]$, then the following statements are equivalent*:

(1) $G \in OC^\circ$ *and* OM° *on* $[a, b]$, *and*

(2) $\ln(1+G) \in OA^\circ$ *on* $[a, b]$.

Furthermore, ${}_a\prod^b (1+G) = \exp \int_a^b \ln(1+G)$.

PROOF. (1)→(2) It follows from Theorem 1 that $\int_a^b \ln(1+G)$ exists. Hence, it is only necessary to show that

$$\int_a^b \left| \ln(1 + G) - \int \ln(1 + G) \right| = 0.$$

Note that $G \in OP^\circ$ and OU° on $[a, b]$ [Lemma 1]. Let $\varepsilon > 0$. There exists a subdivision D of $[a, b]$ and a number B such that if $J = \{x_q\}_0^n$ is a refinement of D and J_q is a subdivision of $[x_{q-1}, x_q]$ for $1 \leqq q \leqq n$, then

(1) $|[\prod_r^s (1+G_q)]^{-1}| < B$ for $1 \leqq r \leqq s \leqq n$,

(2) $|1 + \sum_{i=1}^\infty (-1)^{i-1}\{[1+G_q][\prod_{J_q(I)} (1+G)]^{-1} - 1\}^{i-1}/i| < B$ for $1 \leqq q \leqq n$,

(3) $|[1+G_q][\prod_{J_q(I)} (1+G)]^{-1} - 1| < 1$ for $1 \leqq q \leqq n$, and

(4) $\sum_{q=1}^n |1 + G_q - \prod_{J_q(I)} (1+G)| < \varepsilon/2B^2$.

Let $J = \{x_q\}_0^n$ be a refinement of D. For $1 \leqq q \leqq n$, let J_q be a subdivision of $[x_{q-1}, x_q]$ such that

$$\left| \sum_{J_q(I)} \ln(1 + G) - \int_{x_{q-1}}^{x_q} \ln(1 + G) \right| < \varepsilon/2n.$$

Thus,

$$\begin{aligned}
\sum_{q=1}^{n} &\left| \ln(1+G_q) - \int_{x_{q-1}}^{x_q} \ln(1+G) \right| \\
&\leqq \sum_{q=1}^{n} \left| \ln\left\{ [1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1}\left[\prod_{J_q(I)} (1+G)\right]\right\} - \sum_{J_q(I)} \ln(1+G) \right| \\
&\quad + \sum_{q=1}^{n} \left| \sum_{J_q(I)} \ln(1+G) - \int_{x_{q-1}}^{x_q} \ln(1+G) \right| \\
&< \sum_{q=1}^{n} \left| \ln\left\{ [1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1}\right\}\right| + n(\varepsilon/2n) \\
&= \sum_{q=1}^{n} \left| \sum_{i=1}^{\infty} (-1)^{i-1}\left\{ [1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1} - 1\right\}^{i} \Big/ i \right| + \varepsilon/2 \\
&\leqq \sum_{q=1}^{n} \left| [1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1} - 1 \right| \\
&\qquad \cdot \left| 1 + \sum_{i=1}^{\infty} (-1)^{i-1}\left\{ [1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1} - 1\right\}^{i-1} \Big/ i \right| + \varepsilon/2 \\
&\leqq B\sum_{q=1}^{n} \left| [1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1} - 1 \right| + \varepsilon/2 \\
&\leqq B\sum_{q=1}^{n} \left| 1+G_q - \prod_{J_q(I)} (1+G) \right| \left| \left[\prod_{J_q(I)} (1+G)\right]^{-1} \right| + \varepsilon/2 \\
&\leqq B^2\sum_{q=1}^{n} \left| 1+G_q - \prod_{J_q(I)} (1+G) \right| + \varepsilon/2 \\
&< B^2(\varepsilon/2B^2) + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, $\ln(1+G) \in OA^\circ$ on $[a, b]$.

PROOF. (2)→(1) It follows from Theorem 1 that

(1) $G \in OC^\circ$ on $[a, b]$,

(2) ${}_x\prod^y (1+G)$ exists for $a \leqq x < y \leqq b$, and ${}_a\prod^b (1+G) = \exp\int_a^b \ln(1+G)$.

Hence, it is only necessary to show that

$$\int_a^b \left| 1 + G - \prod (1+G) \right| = 0.$$

Let $\varepsilon > 0$. Lemma 1 implies that $G \in OP^\circ$ and OU° on $[a, b]$. There exists a subdivision D of $[a, b]$ and a number B such that if $J = \{x_q\}_0^n$ is a refinement of D and J_q is a subdivision of $[x_{q-1}, x_q]$ for $1 \leqq q \leqq n$, then

(1) $|\prod_r^s (1+G_q)| < B$ for $1 \leqq r \leqq s \leqq n$,

(2) $|[1+G_q][\prod_{J_q(I)} (1+G)]^{-1} - 1| < 1$ for $1 \leqq q \leqq n$,

(3) $|\{P_q\}^{-1}|<B$ for $1\leqq q\leqq n$, where

$$P_q = 1 + \sum_{i=2}^{\infty} (-1)^{i-1}\left\{[1+G_q]\left[\prod_{J_q(I)} (1+G)\right]^{-1} - 1\right\}^{i-1}\Big/ i,$$

and

(4) $\sum_{q=1}^{n} |\ln(1+G_q)-\sum_{J_q(I)} \ln(1+G)|<\varepsilon/2B^2$.

Let $J=\{x_q\}_0^n$ be a refinement of D. Further, for $1\leqq q\leqq n$, let J_q be a subdivision of $[x_{q-1}, x_q]$ such that

$$\left|\prod_{J_q(I)}(1+G) - {}_{x_{q-1}}\prod^{x_q} (1+G)\right| < \varepsilon/2n.$$

Thus,

$$\begin{aligned}
&\sum_{q=1}^{n}\left|1+G_q - {}_{x_{q-1}}\prod^{x_q}(1+G)\right| \\
&\quad < \sum_{q=1}^{n}\left|1+G_q - \prod_{J_q(I)}(1+G)\right| + n(\varepsilon/2n) \\
&\quad \leqq \sum_{q=1}^{n}\left|[1+G_q]\left[\prod_{J_q(I)}(1+G)\right]^{-1} - 1\right|\left|\prod_{J_q(I)}(1+G)\right| + \varepsilon/2 \\
&\quad \leqq B\sum_{q=1}^{n}\left|[1+G_q]\left[\prod_{J_q(I)}(1+G)\right]^{-1} - 1\right| + \varepsilon/2 \\
&\quad \leqq B\sum_{q=1}^{n}\left|\left\{[1+G_q]\left[\prod_{J_q(I)}(1+G)\right]^{-1} - 1\right\}\{P_q\}\right| |\{P_q\}^{-1}| + \varepsilon/2 \\
&\quad \leqq B^2\sum_{q=1}^{n}\left|\sum_{i=1}^{\infty}(-1)^{i-1}\left\{[1+G_q]\left[\prod_{J_q(I)}(1+G)\right]^{-1} - 1\right\}^{i}\Big/ i\right| + \varepsilon/2 \\
&\quad = B^2\sum_{q=1}^{n}\left|\ln\left\{[1+G_q]\left[\prod_{J_q(I)}(1+G)\right]^{-1}\right\}\right| + \varepsilon/2 \\
&\quad = B^2\sum_{q=1}^{n}\left|\ln[1+G_q] - \prod_{J_q(I)}\ln(1+G)\right| + \varepsilon/2 \\
&\quad < B^2(\varepsilon/2B^2) + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, $G\in OM^\circ$ on $[a, b]$.

LEMMA 2. *If G is a function from $R\times R$ to X, $G\in OB^\circ$ on $[a, b]$, $\{S_n\}_1^\infty$ is a sequence of functions from $R\times R$ to X converging uniformly to a bounded function S on $[a, b]$ and $S_nG\in OA^\circ$ on $[a, b]$ for $n=1, 2, \cdots$, then $SG\in OA^\circ$ on $[a, b]$, $\lim_{n\to\infty}\int_a^b S_nG$ exists and $\int_a^b SG=\lim_{n\to\infty}\int_a^b S_nG$.*

The proof of Lemma 2 is straightforward and, therefore, we omit it.

LEMMA 3. *If H and G are functions from $R\times R$ to X, $H\in 0L^\circ$ on $[a, b]$ and $G\in OA^\circ$ and OB° on $[a, b]$, then $HG\in OA^\circ$ and OM° on $[a, b]$* [**3**, *Theorem* 2, *p*. 494].

LEMMA 4. *If $\beta>0$ and G is a function from $R\times R$ to X such that $|G|<1-\beta$ on $[a, b]$, $G\in OL^\circ$ on $[a, b]$ and $G^2\in OA^\circ$ and OB° on $[a, b]$, then*

$$\sum_{n=2}^{\infty}(-1)^{n-1}G^n/n$$

is in OA° on $[a, b]$.

PROOF. Since $G\in OL^\circ$ on $[a, b]$,

$$\sum_{n=2}^{p}(-1)^{n-1}(G^{n-2})/n$$

is in OL° on $[a, b]$ for $p=2, 3, 4, \cdots$. Thus,

$$\left[\sum_{n=2}^{p}(-1)^{n-1}(G^{n-2})/n\right]G^2$$

is in OA° on $[a, b]$ for $p=2, 3, 4, \cdots$ [Lemma 3]. Therefore, since

$$\left\{\sum_{n=2}^{p}(-1)^{n-1}(G^{n-2})/n\right\}_{p=2}^{\infty}$$

converges uniformly to

$$\sum_{n=2}^{\infty}(-1)^{n-1}(G^{n-2})/n$$

on $[a, b]$, it follows from Lemma 2 that

$$\left[\sum_{n=2}^{\infty}(-1)^{n-1}(G^{n-2})/n\right]G^2=\sum_{n=2}^{\infty}(-1)^{n-1}G^n/n$$

is in OA° on $[a, b]$.

We note that a similar justification of Lemma 4 for real valued functions is contained in the proof of Theorem 5 (1→2) of a previous paper by the author [**4**].

THEOREM 3. *Suppose X is commutative. If $\beta>0$ and G is a function from $R\times R$ to X such that $|G|<1-\beta$ on $[a, b]$, $G\in OL^\circ$ on $[a, b]$ and $G^2\in OA^\circ$ and OB° on $[a, b]$, then the following statements are equivalent:*

(1) $G\in OC^\circ$ on $[a, b]$, *and*

(2) $\int_a^b G$ *exists.*

Furthermore, ${}_a\prod^b(1+G)=\exp\int_a^b\ln(1+G)$.

PROOF. It follows from Lemma 4 that

$$\int_a^b \sum_{n=2}^{\infty}(-1)^{n-1}G^n/n$$

exists. Hence, if (1) is true, then $\int_a^b G$ exists since Theorem 1 implies that

$$\int_a^b \ln(1+G) = \int_a^b \sum_{n=1}^{\infty}(-1)^{n-1}G^n/n$$

exists. Also, if (2) is true, then Theorem 1 implies that (1) is true since

$$\int_a^b \sum_{n=1}^{\infty}(-1)^{n-1}G^n/n = \int_a^b \ln(1+G)$$

exists. In addition, Theorem 1 also implies that

$${}_a\prod{}^b (1+G) = \exp\int_a^b \ln(1+G).$$

THEOREM 4. *Suppose X is commutative. If $\beta>0$ and G is a function from $R\times R$ to X such that $|G|<1-\beta$ on $[a,b]$, $G\in OL^\circ$ on $[a,b]$ and $G^2\in OA^\circ$ and OB° on $[a,b]$, then the following statements are equivalent:*

(1) *$G\in OC^\circ$ and OM° on $[a,b]$, and*

(2) *$G\in OA^\circ$ on $[a,b]$.*

Furthermore, ${}_a\prod^b (1+G)=\exp\int_a^b \ln(1+G)$.

PROOF. It follows from Lemma 4 that

$$\sum_{n=2}^{\infty}(-1)^{n-1}G^n/n$$

is in OA° on $[a,b]$. Hence, if (1) is true, then $G\in OA^\circ$ on $[a,b]$ since Theorem 2 implies that

$$\ln(1+G) = \sum_{n=2}^{\infty}(-1)^{n-1}G^n/n$$

is in OA° on $[a,b]$. Also, if (2) is true, then Theorem 2 implies that (1) is true since

$$\sum_{n=1}^{\infty}(-1)^{n-1}G^n/n = \ln(1+G)$$

is in OA° on $[a,b]$. In addition, Theorem 2 also implies that

$$\prod{}^b (1+G) = \exp\int_a^b \ln(1+G).$$

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