

AN EXISTENCE THEOREM FOR SUM AND PRODUCT INTEGRALS

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ABSTRACT. Functions are from $S \times S$ to N , where S and N denote a linearly ordered set and a normed ring, respectively.

THEOREM. If $\{a, b\} \in S \times S$, G has bounded variation on $\{a, b\}$, $\{F_n\}_1^\infty$ converges uniformly to a bounded function F on $\{a, b\}$ and either

$$(1) \quad x \prod^y (1 + F_n G) \quad \text{and} \quad \int_a^b |1 + F_n G - \prod (1 + F_n G)|$$

exist for $n=1, 2, \dots$ and each subdivision $\{a, x, y, b\}$ of $\{a, b\}$ and $\{\int_a^b |1 + F_n G - \prod (1 + F_n G)|\}_1^\infty$ converges to zero or

$$(2) \quad \int_a^b F_n G \quad \text{and} \quad \int_a^b |F_n G - \int F_n G|$$

exist for $n=1, 2, \dots$ and $\{\int_a^b |F_n G - \int F_n G|\}_1^\infty$ converges to zero, then (conclusion) ${}_a \prod^b (1 + FG)$, ${}_a \prod^b (1 + |FG|)$, $\int_a^b FG$ and $\int_a^b |FG|$ exist. Further, $\int_a^b |1 + FG - \prod (1 + FG)|$, $\int_a^b |1 + |FG| - \prod (1 + |FG|)|$, $\int_a^b |FG - \int FG|$ and $\int_a^b \|FG| - \int \|FG\|$ exist and are zero.

All integrals and definitions are of the subdivision-refinement type, and functions are from $S \times S$ to N , where S denotes a linearly ordered set and N denotes a ring which has a multiplicative identity element represented by 1 and has a norm $|\cdot|$ with respect to which N is complete and $|1|=1$. The function $G \in OM^\circ$ on $\{a, b\}$ only if ${}_x \prod^y (1 + G)$ exists for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$ and $\int_a^b |1 + G - \prod (1 + G)| = 0$, and $G \in OA^\circ$ on $\{a, b\}$ only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$. If $J = \{x_q\}_0^n$ is a subdivision of $\{a, b\}$, then $G_q = G(x_{q-1}, x_q)$. Similarly, if F_n is a function, then $F_{nq} = F_n(x_{q-1}, x_q)$. Further, we use $J(I)$, $\prod_{J(I)} (1 + G)$ and $\sum_{J(I)} G$ to represent $\{(x_{q-1}, x_q)\}_1^n$, $\prod_1^n (1 + G_q)$ and $\sum_1^n G_q$, respectively. In addition, G is bounded on $\{a, b\}$, $G \in OP^\circ$ on $\{a, b\}$, and $G \in OB^\circ$ on $\{a, b\}$ only if there exists a subdivision D of $\{a, b\}$ and a number B such that if $J = \{x_q\}_0^n$ is a refinement of D , then

- (1) $|G(u)| < B$ for $u \in J(I)$,
- (2) $|\prod_1^j (1 + G_q)| < B$ for $1 \leq i \leq j \leq n$, and
- (3) $\sum_{J(I)} |G| < B$,

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respectively. Note that $OB^\circ \subseteq OP^\circ$. The statement that the sequence of functions $\{F_n\}_1^\infty$ converges uniformly to a function F on $\{a, b\}$ means if $\varepsilon > 0$ then there exists a subdivision D of $[a, b]$ and a positive integer N such that if J is a refinement of D and $n > N$, then $|F(u) - F_n(u)| < \varepsilon$ for $u \in J(I)$. Consult B. W. Helton [2] and J. S. MacNerney [4] for additional details.

LEMMA 1. *If $\{a, b\} \in S \times S$ and $F \in OP^\circ$ and $G \in OP^\circ$ on $\{a, b\}$, then there exists a subdivision D of $\{a, b\}$ and a number B such that if $J = \{x_q\}_0^r$ is a refinement of D and $1 \leq i \leq j \leq r$, then*

$$\left| \prod_i^j (1 + F_q) - \prod_i^j (1 + G_q) \right| \leq B \sum_i^j |F_q - G_q|$$

[2, Theorem 3.2, p. 300].

LEMMA 2. *If $\{a, b\} \in S \times S$, $G \in OB^\circ$ on $\{a, b\}$, $\{F_n\}_1^\infty$ converges uniformly to a bounded function F on $\{a, b\}$ and ${}_a\prod^b (1 + F_n G)$ exists for $n = 1, 2, \dots$, then*

- (1) $\lim_{n \rightarrow \infty} {}_a\prod^b (1 + F_n G)$ exists, and
- (2) ${}_a\prod^b (1 + FG)$ exists and is $\lim_{n \rightarrow \infty} {}_a\prod^b (1 + F_n G)$.

PROOF. We first show that (1) is true. Let $\varepsilon > 0$. There exists a subdivision D_1 of $\{a, b\}$, a number B and a positive integer N such that if J is a refinement of D_1 and $n \geq m > N$, then

- (1) $\sum_{J(I)} |G| < B$,
- (2) $|\prod_{J(I)} (1 + F_n G) - \prod_{J(I)} (1 + F_m G)| \leq B \sum_{J(I)} |F_n G - F_m G|$, and
- (3) $|F_n(u) - F_m(u)| < \varepsilon / 2B^2$ for $u \in J(I)$.

Let $n \geq m > N$. There exists a subdivision D_2 of $\{a, b\}$ such that if J is a refinement of D_2 , then

- (1) $|{}_a\prod^b (1 + F_n G) - \prod_{J(I)} (1 + F_n G)| < \varepsilon / 4$, and
- (2) $|\prod_{J(I)} (1 + F_m G) - {}_a\prod^b (1 + F_m G)| < \varepsilon / 4$.

Let $D = D_1 \cup D_2$, and suppose J is a refinement of D . Thus,

$$\begin{aligned} & \left| {}_a\prod^b (1 + F_n G) - {}_a\prod^b (1 + F_m G) \right| \\ & < \left| \prod_{J(I)} (1 + F_n G) - \prod_{J(I)} (1 + F_m G) \right| + \varepsilon / 2 \\ & \leq B \sum_{J(I)} |F_n G - F_m G| + \varepsilon / 2 < B(\varepsilon / 2B^2)B + \varepsilon / 2 = \varepsilon. \end{aligned}$$

We now show that ${}_a\prod^b (1 + FG)$ exists and is L , where

$$L = \lim_{n \rightarrow \infty} {}_a\prod^b (1 + F_n G).$$

Let $\varepsilon > 0$. There exists a subdivision D of $\{a, b\}$, a number B and a positive integer n such that if J is a refinement of D , then

- (1) $\sum_{J(I)} |G| < B$,
- (2) $|\prod_{J(I)} (1 + FG) - \prod_{J(I)} (1 + F_n G)| \leq B \sum_{J(I)} |FG - F_n G|$,
- (3) $|\prod_{J(I)} (1 + F_n G) - L| < \varepsilon/2$, and
- (4) $|F(u) - F_n(u)| < \varepsilon/2B^2$ for $u \in J(I)$.

Thus, if J is a refinement of D , then

$$\begin{aligned} \left| \prod_{J(I)} (1 + FG) - L \right| &< \left| \prod_{J(I)} (1 + FG) - \prod_{J(I)} (1 + F_n G) \right| + \varepsilon/2 \\ &\leq B \sum_{J(I)} |FG - F_n G| + \varepsilon/2 < B(\varepsilon/2B^2)B + \varepsilon/2 = \varepsilon. \end{aligned}$$

LEMMA 3. If $\{a, b\} \in S \times S$, $G \in OB^\circ$ on $\{a, b\}$, $\{F_n\}_1^\infty$ converges uniformly to a bounded function F on $\{a, b\}$ and $\int_a^b F_n G$ exists for $n=1, 2, \dots$, then

- (1) $\lim_{n \rightarrow \infty} \int_a^b F_n G$ exists, and
- (2) $\int_a^b FG$ exists and is $\lim_{n \rightarrow \infty} \int_a^b F_n G$.

INDICATION OF PROOF. Lemma 3 follows by an argument that is similar to the one used for Lemma 2.

LEMMA 4. If $\{a, b\} \in S \times S$ and $G \in OB^\circ$ on $\{a, b\}$, then the following statements are equivalent:

- (1) $G \in OA^\circ$ on $\{a, b\}$, and
- (2) $G \in OM^\circ$ on $\{a, b\}$ [2, Theorem 3.4, p. 301].

LEMMA 5. If $\{a, b\} \in S \times S$ and G is a real valued function such that $\int_a^b G$ exists, then $G \in OA^\circ$ on $\{a, b\}$.

Lemma 5 is due to A. Kolmogoroff [3, p. 669]. The reader is also referred to results by W. D. L. Appling [1, Theorems 1, 2, p. 155] and B. W. Helton [2, Theorem 4.1, p. 304].

LEMMA 6. If $\{a, b\} \in S \times S$, $G \in OB^\circ$ on $\{a, b\}$ and either $G \in OA^\circ$ on $\{a, b\}$ or $G \in OM^\circ$ on $\{a, b\}$, then $|G| \in OA^\circ$ and OM° on $\{a, b\}$.

PROOF. It follows from Lemma 4 that $G \in OA^\circ$ on $\{a, b\}$. There exists a number B such that if J is a subdivision of $\{a, b\}$ then $\sum_{J(I)} |\int G| < B$. Let V be the set such that $v \in V$ only if there exists a subdivision J of $\{a, b\}$ such that $v = \sum_{J(I)} |\int G|$. The set V is bounded and thus has a least upper bound L .

We now show that $\int_a^b |G|$ exists and is L . Let $\varepsilon > 0$. There exists a subdivision E of $\{a, b\}$ such that if J is a refinement of E , then

- (1) $\sum_{J(I)} |\int G| > L - \varepsilon/2$, and
- (2) $\sum_{J(I)} |G - \int G| < \varepsilon/2$.

Suppose J is a subdivision of E . Thus,

$$\sum_{J(I)} |G| \leq \sum_{J(I)} \left| \int G \right| + \sum_{J(I)} \left| G - \int G \right| < L + \varepsilon/2,$$

and

$$\sum_{J(I)} |G| > \sum_{J(I)} \left| \int G \right| - \sum_{J(I)} \left| G - \int G \right| > (L - \varepsilon/2) - \varepsilon/2.$$

Therefore,

$$L - \varepsilon < \sum_{J(I)} |G| < L + \varepsilon.$$

Hence, $\int_a^b |G|$ exists. Thus, it follows from Lemma 5 that $|G| \in OA^\circ$ on $\{a, b\}$ and from Lemma 4 that $|G| \in OM^\circ$ on $\{a, b\}$.

THEOREM. *If $\{a, b\} \in S \times S, G \in OB^\circ$ on $\{a, b\}, \{F_n\}_1^\infty$ converges uniformly to a bounded function F on $\{a, b\}$ and either*

(1) $\int_a^b (1 + F_n G)$ and $\int_a^b |1 + F_n G - \int (1 + F_n G)|$ exist for $n=1, 2, \dots$ and each subdivision $\{a, x, y, b\}$ of $\{a, b\}$ and $\{\int_a^b |1 + F_n G - \int (1 + F_n G)|\}_1^\infty$ converges to zero, or

(2) $\int_a^b F_n G$ and $\int_a^b |F_n G - \int F_n G|$ exist for $n=1, 2, \dots$ and $\{\int_a^b |F_n G - \int F_n G|\}_1^\infty$ converges to zero.

Then $FG \in OM^\circ$ and OA° on $\{a, b\}$ and $|FG| \in OM^\circ$ and OA° on $\{a, b\}$.

PROOF. Suppose (1) is true. It follows from Lemma 2 that $\int_a^b (1 + FG)$ exists for each subdivision $\{a, x, y, b\}$ of $\{a, b\}$. We now show that

$$\int_a^b \left| 1 + FG - \int (1 + FG) \right| = 0.$$

Let $\varepsilon > 0$. There exists a subdivision D of $\{a, b\}$, a number B and a positive integer n such that if $J = \{x_q\}_0^r$ is a refinement of D and J_q is a subdivision of $\{x_{q-1}, x_q\}$ for $1 \leq q \leq r$, then

(1) $\sum_{J(I)} |G| < B,$

(2) $|\int_{J_q(I)} (1 + F_n G) - \int_{J_q(I)} (1 + FG)| \leq B \sum_{J_q(I)} |F_n G - FG|,$

(3) $|F(u) - F_n(u)| < \varepsilon/4B$ and $|F_n(u) - F(u)| < \varepsilon/4B^2$ for $u \in J(I)$, and

(4) $\sum_{q=1}^r |1 + F_n G - \int_{J_q(I)} (1 + F_n G)| < \varepsilon/4.$

Let $J = \{x_q\}_0^r$ be a refinement of D . For $q=1, 2, \dots, r$, there exists a subdivision J_q of $\{x_{q-1}, x_q\}$ such that

$$\left| \int_{J_q(I)} (1 + FG) - \int_{x_{q-1}}^{x_q} (1 + FG) \right| < \varepsilon/4r.$$

Thus,

$$\begin{aligned}
 & \sum_{J(I)} \left| 1 + FG - \prod (1 + FG) \right| \\
 & \leq \sum_{J(I)} |F - F_n| |G| + \sum_{q=1}^r \left| 1 + F_{nq} G_q - \prod_{J_q(I)} (1 + FG) \right| \\
 & \quad + \sum_{q=1}^r \left| \prod_{J_q(I)} (1 + FG) - x_{q-1} \prod_{J_q(I)} x_q (1 + FG) \right| \\
 & < (\varepsilon/4B)B + \sum_{q=1}^r \left| 1 + F_{nq} G_q - \prod_{J_q(I)} (1 + FG) \right| + r(\varepsilon/4r) \\
 & \leq \sum_{q=1}^r \left| 1 + F_{nq} G_q - \prod_{J_q(I)} (1 + F_n G) \right| \\
 & \quad + \sum_{q=1}^r \left| \prod_{J_q(I)} (1 + F_n G) - \prod_{J_q(I)} (1 + FG) \right| + \varepsilon/2 \\
 & \leq \varepsilon/4 + \sum_{q=1}^r \left[B \sum_{J_q(I)} |F_n - F| |G| \right] + \varepsilon/2 \\
 & \leq \varepsilon/4 + B(\varepsilon/4B^2)B + \varepsilon/2 = \varepsilon.
 \end{aligned}$$

Therefore, $FG \in OM^\circ$ on $\{a, b\}$, and hence, $FG \in OA^\circ$ on $\{a, b\}$ by Lemma 4. Thus, it follows from Lemma 6 that $|FG| \in OM^\circ$ and OA° on $\{a, b\}$.

Suppose (2) is true. The desired result follows by using Lemma 3 and an argument similar to the one employed in the preceding paragraph.

The question naturally arises as to whether or not there exist functions and sequences satisfying the conditions in the hypothesis of the Theorem. We now supply an example which was suggested by a result of Appling [1, Theorem 2 (1→2), pp. 155–156] on real Hilbert spaces. We consider the set of infinite diagonal matrices with bounded elements and $|M| = \text{lub } |m_{ij}|$. For each positive integer p , let A_p be the infinite diagonal matrix such that $a_{pp} = 1$ and $a_{qq} = 0$ if $q \neq p$. Let $A = \bigcup_{p=1}^\infty A_p$. Let f be a reversible function from the rational numbers contained in $(0, 1)$ to A . There exists a function G of subintervals of $[0, 1]$ such that if $[u, v]$ is a subinterval of $[0, 1]$ then there exists a rational number r contained in $[u, v]$ such that

$$G(u, v) = (v - u)f(r).$$

For each positive integer n , let $F_n(u, v)$ be the diagonal matrix B_n such that $b_{qq} = 1/n$. Thus, if $[u, v]$ is a subinterval of $[0, 1]$, then

$$\int_u^v F_n G = 0 \quad \text{and} \quad \prod_u^v (1 + F_n G) = 1.$$

Hence,

$$\int_0^1 \left| F_n G - \int F_n G \right| = \int_0^1 |F_n G| = 1/n,$$

and

$$\int_0^1 \left| 1 + F_n G - \prod (1 + F_n G) \right| = \int_0^1 |1 + F_n G - 1| = \int_0^1 |F_n G| = 1/n.$$

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