$C^k$, WEAKLY HOLOMORPHIC FUNCTIONS ON ANALYTIC SETS

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Abstract. Let $V$ be a complex analytic set and $p \in V$. Let $\mathcal{O}(V)$, $\mathcal{O}(V)$, and $C^k(V)$ denote respectively the rings of germs of holomorphic, weakly holomorphic, and $k$-times continuously differentiable functions on $V$. Spallek proved that there exists sufficiently large $k$ such that $C^k(V) \cap \mathcal{O}(V) = \mathcal{O}(V)$. In this paper I give a new proof of this result for curves and hypersurfaces which also establishes that the conduction number of the singularity is an upper bound for $k$. This estimate also holds for any pure dimensional variety off of a subvariety of codimension two.

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An element $u \in \mathcal{O}$ is said to be a universal denominator if $u\mathcal{O} \subseteq \mathcal{O}$. Let $I$ be the ideal of $\mathcal{O}$ of all functions vanishing on $\text{Sing}(V)$ and $J$ be the ideal of universal denominators. Then $\text{locus}(J) \subseteq \text{Sing } V$ [3, p. 56], so by the Hilbert Nullstellensatz there is a positive integer $N$ such that $I^N \subseteq J$. The main result of this paper is that $k$ can be chosen so that $k \leq N$.

Siu has proven [5] that if $k(p)$ is the minimal value of $k$ such that $C^k \cap \mathcal{O} = \mathcal{O}$ at the point $p \in V$, then the function $k(p)$ is bounded on compact subsets of $V$. This result also follows from the above estimate, by the coherence of the ideal sheafs of $I$ and $J$. (The ideal sheaf of $J$ is coherent [2, Theorem 22] because it is the kernel of $\mathcal{O} \rightarrow \text{Hom}_G(\mathcal{O}, \mathcal{O}/\mathcal{O})$.)

This estimate is, in general, not the best possible. In an earlier work [1], for the example of a curve in $C^2$ normalized by a map $t \rightarrow (t^p, r^p u(t))$ where $u(t)$ is a unit, $p > q$, and $p$ and $q$ are relatively prime, it was shown that...
k = [(p/q)(q-2)] + 1 and N = [(p/q)(q-1)], where [x] for any real number x is the greatest integer less than or equal to x.

The original estimate for k obtained by Spallek [6] seems a bit obscure in the case of a nonisolated singularity. This is made clearer by Siu in [5] and Spallek in [8]:

Suppose A is an analytic set of pure dimension r, π: A → C^r a branched covering of sheeting order μ, z_{r+1} a direction in C^n which separates the fibers of π almost everywhere, and δ the discriminant of the minimal polynomial in _A over _C; then k ≤ μ(m + 1) where δ(π(z)) ⊆ δ_μ. Now m is related to the conduction number N, but not necessarily equal—depending upon whether the projection π has minimal multiplicity—so the estimate in this paper is better approximately by a factor of the minimal multiplicity. For the above mentioned case of a curve in C^2, direct computation shows that Spallek’s estimate is k = p(q-1) + q.

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1. Suppose V is a complex analytic hypersurface in C^n, the projection π: C^n → C^{n-1} to the first n-1 coordinates gives a q sheeted branched cover of V with branch set B, B' = π(B) and z' = π(z). Now π induces a homomorphism _n→_n making _n(V) into a finitely generated π−1 module with generators 1, z_n, · · · , z_{n-1}. Hence if f ∈ _n(V), then f can be written as ∑_{i=0}^{q-1} b_i(z')z_n^{q-i-1}. For any weakly holomorphic function f, there is a canonical attempted extension to the ambient space, which is in fact holomorphic, if f is holomorphic: for z' ∈ B', let

\[ g(z', z_n) = \sum_{j=1}^{q} \left( \prod_{k \neq j} \frac{z_n - \alpha_k(z')}{\alpha_j(z') - \alpha_k(z')} \right) f(z', \alpha_j(z')) \]

\[ = \sum_{i=0}^{q-1} (-1)^i b_i(z')z_n^{q-i-1}, \]

\[ b_i(z') = \sum_{j=1}^{q} \frac{\sigma_i(z'), \cdots, \hat{\sigma}_i(z'), \cdots, \alpha_q(z')} {\prod_{k \neq j} (\alpha_j(z') - \alpha_k(z'))} f(z', \alpha_j(z')) \]

where hatted terms are deleted, σ_i is the elementary symmetric polynomial of degree i, and {σ_i(z'): 1 ≤ j ≤ q} are the values of z_n on the fiber π^{-1}(z').

For z ∈ V, z_n = some α_i so g(z) = f(z). The coefficients b_i are well defined (do not depend upon the ordering of the α_i's), b_i ∈ _n-B' and g ∈ _n(C^n-B' × C^n-r). By the Riemann removable singularities theorem, g extends holomorphically to C^n if and only if the b_i are bounded near B'. (If g is holomorphic, then so is (∂^{q-1}/∂z_n)g = (q-1)! b_0(z'), etc.)
If \( f \in \mathcal{O}(V) \), then as pointed out by Spallek [5, Abschnitt 6], the Newton Interpolation Formula [4, pp. 10-16] says that if

\[
[f_1, \ldots, f_q] = \sum_{j=1}^{q} \prod_{i \neq k} (\lambda_j(z') - \lambda_k(z'))
\]

then there exists a complex constant \( \lambda \) with \( |\lambda| \leq 1 \) and real numbers \( \delta_1, \ldots, \delta_q \geq 0 \) with \( \sum \delta_i = 1 \) such that

\[
\frac{f(z', \lambda_1(z') \lambda_2(z') \ldots \lambda_q(z'))}{\prod_{i \neq k} (\lambda_i(z') - \lambda_k(z'))}.
\]

Now

\[
s_i(\lambda_1, \ldots, \lambda_q) = s_i(\lambda_1, \ldots, \lambda_q) - s_{i-1}(\lambda_1, \ldots, \lambda_q) = \sum_{l=0}^{i} (-1)^l \sigma_{i-l}(\lambda_1, \ldots, \lambda_q)
\]

so

\[
b_i(z') = \sum_{l=0}^{i} (-1)^l \sigma_{i-l}(\lambda_1, \ldots, \lambda_q) [(z_n^1 f_1, \ldots, (z_n^1 f_q)]
\]

and it follows immediately that \( b_i \) is bounded near \( B' \) by the continuity of \( (\partial^{q-1}/\partial z_n)f \).

2. First we consider the one-dimensional case. Let \( V \) be normalized by a map \( \theta(t) = (t^q, t^{p_1}u_1(t), \ldots, t^{p_n}u_{n-1}(t)) \) where each \( p_i \geq q \) and each \( u_i \) is a holomorphic function with \( u_i(0) \neq 0 \). Let \( f \in C^k(V) \cap \mathcal{O}(V) \) and \( T_{\theta}^k(f) \) be the \( k \)th order Taylor series of \( f \) about the origin; write \( T_{\theta}^k(f) = P_kf + Q_kf \) where \( P_kf \) is a homomorphic polynomial and \( Q_kf \) contains the antiholomorphic terms. It is a standard fact that \( f - T_{\theta}^k(f) = o(|z|^k) \). However even more is true:

**Lemma.** \( (f-P_kf)\theta(t) = o(\theta(t)^k) = o(t^k) \).

This is Lemma 3 of [1] and is also essentially contained in [5, paragraph 2.2].

Let \( h = f - P_kf \); we have that \( h \) is also weakly holomorphic, \( h \) is holomorphic if and only if \( f \) is holomorphic, \( h \) is precisely as differentiable as \( f \) and that \( h(\theta(t)) = o(t^k) \) since \( P_kh \equiv 0 \). Hence \( h/z_1^k \) is weakly holomorphic. Since \( z_1^N \) is a universal denominator, \( h z_1^{N-k} \) is holomorphic; for \( k = N \) we have that \( h \) is holomorphic. Thus \( C^N(V) \cap \mathcal{O}(V) = \mathcal{O}(V) \).

More generally, for a variety \( V \) of pure dim \( r \) in \( C^n \), let

\[
C = \text{Sing}(\text{Sing} V) \cup \{p \in V \ | \ \dim C_4(V, p) > r\} \cup \{p \in V \ | \ \dim C_6(V, p) > r + 1\}
\]
where $C_4(V, p)$ and $C_5(V, p)$ are the fourth and fifth Whitney tangent cones to $V$ at $p$ [10]. Then $C$ is an analytic subset of $V$ of codimension at least two [9, Proposition 3.6] and every $p \in V - C$ has an open neighborhood so that after a local biholomorphic change of coordinates the following hold:

(i) For each irreducible component $V_i$ of $V$, $V_i \cap \text{Sing } V = \text{Sing } V_i = C^{r-1}$ [9, Proposition 2.10, 2.12, and 4.5].

(ii) Each component has a one-to-one nonsingular normalization [9, Proposition 4.2] $\phi: D \to V_i$ given by

$$\phi(t_1, \ldots, t_r) = (t_1, \ldots, t_{r-1}, t_r^q, \phi_{r+1}(t), \ldots, \phi_n(t)),$$

where $q$ is the sheeting order of $\pi| V_i$ and $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_r)$. The branching set of this projection is just $\phi(t_r = 0) = C^{r-1}$.

Now let $\text{Cond}_p(V)$ denote the conduction number of the variety at the point $p$ as defined in the introduction. If $V_i$ is a component of $V$ it is clear that any universal denominator for $V$ is a universal denominator for $V_i$ and since $\text{Sing } V_i = \text{Sing } V$, we have that $\text{Cond}_p(V) \supseteq \text{Cond}_p(V_i)$.

For any fixed $s = (t_1, \ldots, t_{r-1})$ consider the curve $W_s$ in $V_i$ given by $t_r = \phi(s, t_r)$. Since this curve $W_s$ lies in $s \times C^{n-r+1}$, weakly holomorphic functions on $W_s$ extend to weakly holomorphic functions on $V_i$ by ignoring the first $r-1$ variables. Hence any universal denominator for $V_i$ is a universal denominator for $W_s$ and $\text{Cond}_p(V_i) \supseteq \text{Cond}_p(W_s)$.

Suppose $f \in C^k \cap \mathcal{O}$, $k \geq \text{Cond}_p(V)$, and $r = n-1$; recall the canonical extension of $\xi_1, \ldots, \xi_r$, so $b_i(z') \in \mathcal{O}(C^{r} - C^{r-1})$. Since $W_s$ is a hypersurface in $s \times C^2$ and $k \geq \text{Cond}_{p(s, 0)}(W_s)$ for each $s$, by the one-dimensional case we have that $b_i$ is bounded on each line

$$L_s = \{(s, z_r) : z_r \in C\}.$$

We need to conclude that $b_i$ extends holomorphically to $C^r$. To do this consider the Laurent power series expansion of $b_i$ in $C^r - C^{r-1}$:

$$b_i(z') = \sum_{m=-\infty}^{+\infty} a_m(z_1, \ldots, z_{r-1})z_r^m, \quad a_m \in \mathcal{O}(C^{r-1}),$$

$$a_m(z') = \frac{1}{2\pi i} \oint_{\xi = z_r} b_i(z'', \xi) d\xi \xi^{m+1}.$$

Choose an $s \in C^{r-1}$ such that for each $a_m$ which is not identically zero, $a_m(s) \neq 0$; if there are any negative exponents of $z_r$ in the above power series expansion, $b_i$ is not bounded on the line $L_s$ (neither a pole nor an essential singularity is bounded).

So far we have shown that $k \leq \text{Cond}_p(V)$ for $p \in V - C$. By coherence of the ideal sheafs of $I(\text{Sing } V)$ and $J$, if $c \in C$, then $\text{Cond}_c(V) \supseteq \text{Cond}_p(V)$.
for all $p$ near $c$. If $f \in C^k \cap \mathcal{O}$ with $k \geq \text{Cond}_c(V)$ then by the last paragraph $b_i \in \mathcal{O}(C^r-\pi(C))$; but $\dim \pi(C) \leq r-2$ so by Hartog’s theorem [3, p. 59], $b_i \in \mathcal{O}(C^r)$ and $f \in \mathcal{O}(V)$.

3. Even without the assumption about the codimension of $V$, it follows that for pure $r$-dimensional $V$ there exists an analytic subset $W$ of $V$ with $\dim W \leq r-2$ such that for every $p \in V-W$, $k(p) \leq N(p)$. Of course this implies the result of the last section since for algebraic complete intersections, singularities in codim two are removable [11].

Instead of directly exhibiting the holomorphic extension, we must resort to more delicate results in sheaf theory, due to Spallek [8, Satz 3.2].

If $f$ is a weakly holomorphic function on $V$, $1 \leq q \leq r$, and $f$ restricted to each $q$-dimensional parallel section is holomorphic, then there is an analytic subset $B^q$ of $V$ (not depending upon $f$) of dimension at most $r-q-1$ so that $f$ is holomorphic on $V-B^q$.

It was shown in the previous section that by restricting to $V-C$, we have $f$ holomorphic on each $s \times C^{n-r}$ and $q=\dim V \cap (s \times C^{n-r})=1$ so $\dim B^q \leq r-2$ and $f$ is holomorphic on $V-(C \cup B^q)$.

BIBLIOGRAPHY


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