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ON A PROBLEM OF MAHLER IN THE GEOMETRY OF NUMBERS

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ABSTRACT. If K is a convex body in R_n and $K(t)$ is that part of K which satisfies $|x_n| \leqq t$, Mahler [2] has shown that $\Delta K(t)/t$ is a decreasing function of t, where $\Delta(K)$ is the critical determinant of K. We generalise Mahler's result in a way different from that conjectured by him.

1. Let K be a closed, convex body in euclidean n-space R_n, symmetric in the origin 0. For a real and positive number t, denote by $K(t)$ that part of K which satisfies $|x_n| \leqq t$, where x_n is the nth coordinate of a fixed coordinate system. If $V(K)$ denotes the volume of K, it was pointed out by Mahler [2] that $V(K(t))/t$ is a monotone decreasing function of t, and he conjectured the same holds true for the critical determinant of K. This number is defined as the infimum of the determinants of those lattices which do not contain an inner point of K, apart from 0. Mahler proved his conjecture for $n=2$. Bambah [1] showed that the same result holds true for the covering constant of K. This number is defined as the supremum of the determinants of those lattices L, for which the collection $X+K$, $X \in L$, covers R_n. The object of this note is to show these results have a natural extension to n dimensions, along different lines to those conjectured by Mahler, the proof of which depends solely on the simple affine properties of the constants involved.

2. Let $m \leqq n$ be a positive integer, and let C denote any set in R_m. We assume that R_m is embedded in R_n in the usual manner. For a real and positive number t, we denote by $K(t)$ the set given by $K(t)=K \cap (tC \times R_{n-m})$, where $tC \times R_{n-m}=\{(x_1, \cdots, x_n) | (x_1, \cdots, x_m) \in tC\}$. We restrict the set C, throughout, to be such that $K(t)$ has Jordan content for all relevant t.

THEOREM 1. *$V(K(t))/t^m$ is a monotone decreasing function of t.*

Let F be a real-valued function of K such that

(i) *$F(T(K))=|\det T| F(K)$ for every nonsingular linear transformation T,*

(ii) *if $K_1 \subset K_2$, then $F(K_1) \leqq F(K_2)$.*

THEOREM 2. *If $m=n-1$, then $F(K(t))/t^{n-1}$ is a monotone decreasing function of t.*

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Theorem 2 represents an extension of the results of Bambah and Mahler to higher dimensions, since the critical determinant and covering constant both satisfy (i) and (ii).

3. **Proof of Theorem 1.** Let $0<t<t'$, and let T denote the linear transformation given by $x_i \to t'x_i/t$, for $i=1, \cdots, m$, so that $\det T=(t'/t)^m$. Now

$$V(K(t')) = \int_{R_m} V_1(x_1, \cdots, x_m)\, dx_1 \cdots dx_m,$$

and

$$V(T(K(t))) = \int_{R_m} V_2(x_1, \cdots, x_m)\, dx_1 \cdots dx_m$$

where $V_1(x_1, \cdots, x_m)$ is the $n-m$ dimensional volume of the set S_1,

$$S_1 = \{(y_1, \cdots, y_n) \in K(t') \mid y_i = x_i \text{ for } i = 1, \cdots, m\};$$

and $V_2(x_1, \cdots, x_m)$ is the $n-m$ dimensional volume of the set S_2,

$$S_2 = \{(y_1, \cdots, y_n) \in T(K(t)) \mid y_i = x_i \text{ for } i = 1, \cdots, m\}.$$

Theorem 2 will follow if we can show $V(S_1) \leqq V(S_2)$, for all points $(x_1, \cdots, x_m) \in R_m$. The set $T^{-1}(S_2)$ is a translation of S_2, and so has volume $V(S_2)$. Further, it is contained in $K(t)$. Let $V(x_1, \cdots, x_m)$ denote the $n-m$ dimensional volume of the set

$$\{(y_1, \cdots, y_n) \in K \mid y_i = x_i \text{ for } i = 1, \cdots, m\}.$$

As K is symmetric in 0, $V(x_1, \cdots, x_m)=V(-x_1, \cdots, -x_m)$. Therefore, by the Brunn-Minkowski theorem,

$$V(0, \cdots, 0) \geqq V(x_1, \cdots, x_m).$$

There is clearly nothing to prove if S_1 is the empty set, so we may assume that S_1 is not empty, from which it follows

$$S_1 = \{(y_1, \cdots, y_n) \in K \mid y_i = x_i \text{ for } i = 1, \cdots, m\},$$

and

$$T^{-1}(S_2) = \{(y_1, \cdots, y_n) \in K \mid y_i = tx_i/t' \text{ for } i = 1, \cdots, m\}.$$

If $S(0, \cdots, 0)$ denotes the section of K at the origin, namely

$$S(0, \cdots, 0) = \{(y_1, \cdots, y_n) \in K \mid y_i = 0 \text{ for } i = 1, \cdots, m\},$$

then, applying the Brunn-Minkowski theorem to the sets $S(0, \cdots, 0)$, $T^{-1}(S_2)$ and S_1 and using the fact already established that $V(0, \cdots, 0) \geqq V(S_1)$, we obtain $V(T^{-1}(S_2)) \geqq V(S_1)$ and Theorem 1 follows.

4. **Proof of Theorem 2.** Let P denote the tangent plane to K, at that point of the boundary of K for which $x_1=\cdots=x_{n-1}=0$, $x_n=a>0$, say. As the quantities in the theorem are unchanged by linear transformations of determinant 1 that leave the first $n-1$ coordinates fixed, we may assume, after application of such a transformation if necessary, that P is given by $x_n=a$. Let $0<t<t'$ and denote by T the linear transformation given by

$$x_i \to t'x_i/t, \quad \text{for } i = 1, \cdots, n-1.$$

If we can show that

(iii) $K(t') \subset T(K(t))$,

then by (i) and (ii),

$$F(K(t')) \leqq F(T(K(t))) = (t'/t)^{n-1}F(K(t))$$

and the theorem is proved.

We assert that (iii) is true, for let $X \in K(t')$. We claim that the point $T^{-1}(X)$ is in K and therefore also in $K(t)$. Otherwise, with $X=(x_1, \cdots, x_n)$, $T^{-1}(X)=(tx_1/t', \cdots, tx_{n-1}/t', x_n)$ is not in K and, since $0<t/t'<1$, so $(0, \cdots, 0, x_n)$ is not in K. However P being a tacplane to K implies $|x_n|>a$, from which it follows that $X \notin K$, which is impossible. This contradiction proves Theorem 2.

References

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