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ON HIGHER ORDER NONSINGULAR IMMERSIONS OF DOLD MANIFOLDS

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ABSTRACT. In this paper we employ γ-operations and characteristic classes to study nonexistence of higher order nonsingular immersions of a Dold manifold into a Euclidean space.

1. **Introduction.** In [2] and [3], W. F. Pohl and E. A. Feldman have considered higher order tangent bundles of a smooth manifold M and the higher order nonsingular immersion of M into euclidean spaces. In [4], [5], and [6], H. Suzuki obtained some higher order nonimmersion theorems of projective spaces into euclidean spaces or projective spaces by means of characteristic classes, γ-operations and spin operations. In [8] C. Yoshioka obtained complete formulas of Stiefel-Whitney classes of higher order tangent bundles of complex projective spaces and Dold manifolds and he applied his results to higher order nonimmersions of these spaces. The purpose of this paper is to prove a higher order nonimmersion theorem for Dold manifolds, using γ-operations and characteristic classes.

2. **Preliminaries.** Let M be an n-dimensional smooth monifold. Let $\tau_p(M)$ be the pth order tangent bundle of M, then $\tau_p(M)$ is a smooth $\nu(n,p)$-vector bundle, where $\nu(n,p)=\binom{n+p}{p}-1$. Set $\tau_p^0(M)=\nu(n,p)-\tau_p(M)$ in $[KO]^{\sim}(M)$. Let λ^i, γ^i and $g\cdot\dim$ be as defined in [1]. Let $W^p(M)$, $\overline{W}^p(M)$, $W_i^p(M)$ and $\overline{W}_i^p(M)$ be total, dual total, i-dimensional and dual i-dimensional Stiefel-Whitney classes of $\tau_p(M)$, respectively. Let $\subseteq_p$ denote pth order nonsingular immersion and $\nsubseteq_p$ its negative. We have the following theorem.

THEOREM 1 ([4]).
(i) *If* $M\subseteq_p R^{\nu(n,p)+k}$, *then* $\overline{W}_i^p=0$ *for* $i>k\geqq 0$;
(ii) *If* $M\subseteq_p R^{\nu(n,p)+k}$, *then* $W_i^p=0$ *for* $0\geqq k>-i$;
(iii) *If* $M\subseteq_p R^{\nu(n,p)+k}$, *then* $\gamma^i(\tau_p^0(M))=0$ *for* $i>k\geqq 0$;
(iv) *If* $M\subseteq_p R^{\nu(n,p)+k}$, *then* $\gamma^i(-\tau_p^0(M))=0$ *for* $0\geqq k>-i$.

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Let $O^i: KO(M) \to KO(M)$ (or $O^i: K(M) \to K(M)$) $(i=1, 2, \cdots)$ be the symmetric ith power operation which has the following properties [4]:

(i) $O^0 x = 1$,

(ii) $O^1 x = x$,

(iii) $O^i(x+y) = \sum_j^i O^j x \cdot O^{i-j} y$ for $x, y \in KO(M)$. We have

THEOREM 2 ([4]). $\tau_p(M) = O^p(\tau(M)+1) - 1$.

3. **Nonimmersion theorem and its proof.** Let RP_m, CP_n and $P(m, n)$ be m-dimensional real, n-dimensional complex projective spaces and a Dold manifold of type (m, n) respectively. In this section we will prove the higher order nonimmersion theorem for $P(m, n)$.

Let $r: K(M) \to KO(M)$ be the realification. Let ξ and $\tilde{\eta}$ be the canonical line bundles over RP_m and CP_n, respectively. Let ξ and η be the bundles over $P(m, n)$ which are defined in [7]. We have the following:

PROPOSITION 1 ([7]). *There exist a* 1*-plane bundle* ξ *and a* 2*-plane bundle* η *over* $P(m, n)$ *such that*

(i) $i^*\xi = \xi$, $j^*\eta = r(\tilde{\eta})$, $i^*\eta = 1 \oplus \xi$;

(ii) $\xi \otimes \xi = 1$, $\xi \otimes \eta = \eta$

where $i: RP_m \to P(m, n)$, $j: CP_n \to P(m, n)$ *are inclusions.*

REMARK. It is easy to prove that $j^*\xi = 1$.

THEOREM 3 ([7]). $\tau(P(m, n)) \oplus \xi \oplus 2 = (m+1)\xi \oplus (n+1)\eta$.

Now let $i: RP_m \to P(m, n)$, $j: CP_n \to P(m, n)$ be inclusions.

From Theorem 2, Theorem 3 and the natural property of O^i, we have:

THEOREM 4.

$$\text{(i)} \quad i^*\tau_p(P(m, n)) = \sum_{0 < \text{odd } i \leq p} \binom{n+p-i-1}{p-i} \binom{m+n+i}{i} \xi + \sum_{0 \leq \text{even } i \leq p} \binom{n+p-i-1}{p-i} \binom{m+n+i}{i} - 1,$$

$$\text{(ii)} \quad j^*\tau_p(P(m, n)) = \sum_{i=0}^{p} \binom{m+p-i-2}{p-i} O^i((n+1)r(\tilde{\eta})) - 1.$$

From Theorem 4 we have,

$$i^*\tau_p^0(P(m, n)) = -\sum_{0 < \text{odd } i \leq p} \binom{n+p-i-1}{p-i} \binom{m+n+i}{i} x$$

where $x = \xi - 1$. By Atiyah [1], we obtain

$$\gamma^i(i^*\tau_p^0(P(m, n))) = \pm 2^{i-1} \binom{A+i-1}{i} x$$

and

$$\gamma^i(-i^*\tau_p^0(P(m, n))) = \pm 2^{i-1}\binom{A}{i}x,$$

where

$$A = \sum_{0<\text{odd } i\leqq p}\binom{n+p-i-1}{p-i}\binom{m+n+i}{i}.$$

Hence

$$\gamma^i(i^*\tau_p^0(P(m, n))) = 0 \Leftrightarrow 2^{i-1}\binom{A+i-1}{i} \equiv 0 \bmod 2^{\phi(m)}$$

and

$$\gamma^i(-i^*\tau_p^0(P(m, n))) = 0 \Leftrightarrow 2^{i-1}\binom{A}{i} \equiv 0 \bmod 2^{\phi(m)}$$

where $\phi(m)$ is defined as the number of integers t with $0<t\leqq m$ and $t\equiv 0, 1, 2$, or $4 \bmod 8$. By using the same method as in [8], we can obtain

$$W_i(j^*\tau_p(P(m, n))) = \binom{B}{i}\beta^i$$

and

$$\overline{W}_i(j^*\tau_p(P(m, n))) = \binom{B+i-1}{i}\beta^i$$

where β is the generator of $H^2(CP_n; Z_2)$ and

$$B = \frac{1}{2}\sum_{0<\text{odd } i\leqq p}\binom{m+p-i-2}{p-i}\binom{2n+i+1}{i}.$$

Let us now define

$$a_1 = \max\left\{i \mid i > 0,\ 2^{i-1}\binom{A+i-1}{i} \not\equiv 0 \bmod 2^{\phi(m)}\right\},$$

$$a_2 = \max\left\{i \mid i > 0,\ 2^{i-1}\binom{A}{i} \not\equiv 0 \bmod 2^{\phi(m)}\right\},$$

$$a_1' = \max\left\{i \mid 0 < i \leqq m,\ \binom{A+i-1}{i} \not\equiv 0 \bmod 2\right\},$$

$$a_2' = \max\left\{i \mid 0 < i \leqq m,\ \binom{A}{i} \not\equiv 0 \bmod 2\right\},$$

$$b_1 = \max\left\{i \mid 0 < i \leqq n,\ \binom{B+i-1}{i} \not\equiv 0 \bmod 2\right\},$$

$$b_2 = \max\left\{i \mid 0 < i \leqq n,\ \binom{B}{i} \not\equiv 0 \bmod 2\right\},$$

$$\sigma_1 = \max\{a_1, a_1', b_1\},$$
$$\sigma_2 = \max\{a_2, a_2', b_2\}.$$

THEOREM 5. *If* $-\sigma_2<k<\sigma_1$, *then* $P(m, n) \not\subseteq_p R^{\nu(m+2n,p)+k}$.

PROOF. By the natural properties of γ^i-operations and Stiefel-Whitney classes, we have that

$$\gamma^i(i^*\tau_p^0(P(m, n))) \neq 0, \qquad \gamma^i(-i^*\tau_p^0(P(m, n))) \neq 0,$$

$W_i(j^*\tau_p(P(m, n)))\not\equiv 0$ and $\overline{W}_i(j^*\tau_p(P(m, n)))\not\equiv 0$ imply

$$\gamma^i(\tau_p^0(P(m, n))) \neq 0, \qquad \gamma^i(-\tau_p^0(P(m, n))) \neq 0,$$

$W_i^p(P(m, n))\not\equiv 0$ and $\overline{W}_i^p(P(m, n))\not\equiv 0$ respectively. Thus Theorem 5 follows from Theorem 1.

REMARKS.

(I) In Theorem 5, if we use Pontrjagin classes instead of Stiefel-Whitney classes, then when $p=1$ we recover the main results of J. J. Ucci [7].

(II) In [8] C. Yoshioka obtained the following formula

$$W^p(P(m, n))=(1+c)^{A'}(1+c+d)^{B'},$$

where c, d are the classes which are defined in [7]. From this formula he obtained the following results.

THEOREM 6. *Let*

$$b_1' = \max\Bigg\{i \mid 0 < i = \alpha + 2\beta \leqq m + 2n, \sum_{0\leqq\gamma\leqq\min}(\alpha, B' - \beta)\binom{A'}{\alpha - \gamma}\frac{B'!}{(B' - \beta - \gamma)!\,\gamma!\,\beta!} \not\equiv 0 \bmod 2\Bigg\},$$

$$b_2' = \max\Bigg\{i \mid 0 < i = \alpha + 2\beta \leqq m + 2n, \sum_{0\leqq\gamma\leqq\min}(\alpha, 2^t - B' - \beta)\binom{A' - 1 + \alpha - \gamma}{\alpha - \gamma} \times \frac{(2^t - B')!}{(2^t - B' - \beta - \gamma)!\,\gamma!\,\beta!} \not\equiv 0 \bmod 2\Bigg\},$$

where

$$A' = \frac{1}{2}\sum_{0\leqq\text{even } i\leqq p}\left\{\binom{2n + i + 1}{i} + (-1)^{p-1}\frac{m + 2(p - i) - 1}{m - 1}\binom{n + 2^{-1}i}{2^{-1}i}\right\} \times \binom{m + p - i - 2}{m - 2},$$

$$B' = \frac{1}{2}\sum_{0<\text{odd } i\leqq p}\binom{2n + i + 1}{i}\binom{m + p - i - 2}{m - 2},$$

t is an integer such that $2^t > \max\{m, n, B'-1\}$. If k is an integer such that $-b_1' < k < b_2'$, then $P(m, n) \not\subseteq_p R^{\nu(m+2n,p)+k}$.

Here we will give some examples to show that in some cases our Theorem 5 can give sharper nonimmersion results than the above theorem.

(1) When $p=1$, then $A'=m$, $B'=n+1$, $A=m+n+1$, $B=n+1$. Let $(m, n)=(14, 1)$. Then $W(P(m, n))=(1+c)^{14}(1+c+d)^2=1$ and Theorem 6 gives no information. By direct calculations we have $\sigma_1=4$. So we have:

COROLLARY 1. $P(14, 1) \not\subseteq R^{16+k}$, $k \leqq 3$.

In general, let $(m, n)=(2^t-2^s, 2^s-1)$, $t \geqq s \geqq 0$. Then $W(P(m, n))=(1+c)^{2^t-2^s}(1+c+d)^{2^s}=1$ and Theorem 6 gives no information. By direct calculations we have $\sigma_1 \geqq 2^{s-1}(2^{t-s}-1)-t$ if $s \geqq 1$, $\sigma_1 \geqq 2^{t-2}$ if $t \geqq 4$ and $s=0$. So we have:

COROLLARY 2. (a) *If $k < 2^{s-1}(2^{t-s}-1)-t$, then $P(m, n) \not\subseteq R^{m+2m+k}$ where $(m, n)=(2^t-2^s, 2^s-1)$, $t \geqq s \geqq 1$.*

(b) ([**1**]). *If $k < 2^{t-2}$, then $P(m, 0) \not\subseteq R^{m+k}$, where $m=2^t-1$, $t \geqq 4$.*

(2) When $p=2$, then $A'=(n+1)^2-m$, $B'=(n+1)(m-1)$, $A=n(m+n+1)$. Let $(m, n)=(12, 3)$. Then

$$W^2(P(m, n)) = (1 + c)^4(1 + c + d)^{44} = 1$$

and Theorem 6 gives no information. By direct calculations we have $\sigma_1=4$, $\sigma_2=4$. Thus we obtain:

COROLLARY 3. $P(12, 3) \not\subseteq_2 R^{189+k}$, $-3 \leqq k \leqq 3$.

In general, let $(m, n)=(2^t-2^s, 2^s-1)$, $t \geqq s \geqq 1$. Then $W^2(P(m, n))=(1+c)^{2^s-2^t+2^s}(1+c+d)^{2^s(2^t-2^s-1)}=(1+c)^{2^t(2^s-1)}=1$ and Theorem 6 gives no information. By direct calculations we have $\sigma_1 \geqq 2^{s-1}(2^{t-s}-1)-t$, $\sigma_2 \geqq 2^{s-1}(2^{t-s}-1)-t$. Thus we obtain:

COROLLARY 4. *If $-2^{s-1}(2^{t-s}-1)+t < k < 2^{s-1}(2^{t-s}-1)-t$, then*

$$P(m, n) \not\subseteq_2 R^{\nu(m+2n,2)+k}$$

where $(m, n)=(2^t-2^s, 2^s-1)$, $t \geqq s \geqq 1$.

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