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# A NOTE ON INDICATOR-FUNCTIONS

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ABSTRACT. A system has the existence-property for abstracts (existence property for numbers, disjunction-property) if whenever $\vdash(\exists x)A(x)$, $\vdash A(\mathrm{t})$ for some abstract (t) ($\vdash A(\boldsymbol{n})$ for some numeral $\boldsymbol{n}$; if whenever $\vdash A \vee B$, $\vdash A$ or $\vdash B$. $(\exists x)A(x)$, $A$, $B$ are closed). We show that the existence-property for numbers and the disjunction property are never provable in the system itself; more strongly, the (classically) recursive functions that encode these properties are not provably recursive functions of the system. It is however possible for a system (e.g., $\mathbf{ZF}+V=L$) to prove the existence-property for abstracts for itself.

In [1], I presented an intuitionistic form $\mathbf{Z}$ of Zermelo-Frankel set-theory (without choice and with weakened regularity) and proved for it the *disjunction-property* (if $\vdash A \vee B$ (closed), then $\vdash A$ or $\vdash B$), the *existence-property* (if $\vdash(\exists x)A(x)$ (closed), then $\vdash A(\mathrm{t})$ for a (closed) comprehension term t) and the *existence-property for numerals* (if $\vdash(\exists x \in \omega)A(x)$ (closed), then $\vdash A(\boldsymbol{n})$ for a numeral $\boldsymbol{n}$). In the appendix to [1], I enquired if these results could be proved constructively; in particular if we could find *primitive recursively* from the proof of $A \vee B$ whether it was $A$ or $B$ that was provable, and likewise in the other two cases.

Discussion of this question is facilitated by introducing the notion of *indicator-functions* in the sense of the following

DEFINITION. Let $\mathbf{T}$ be a consistent theory which contains Heyting arithmetic (possibly by relativization of quantifiers). Then (where $f_\vee$, $f_\exists, f_{\exists\omega}: \omega \to \omega$)

$f_\vee$ is an indicator-function for disjunction $\equiv$ for all $n$, $f_\vee(n)$ is 0 or 1, and if $n$ is (the Gödel-number of) a proof of $A \vee B$, then $f_\vee(n)=0$ implies $\vdash A$ while $f_\vee(n)=1$ implies $\vdash B$;

$f_\exists$ is an indicator-function for existence $\equiv$ for all $n$, if $n$ is a proof of $(\exists x)A(x)$, then $f_\exists(n)$ is the Gödel-number of a term t for which $\vdash A(\mathrm{t})$; and

$f_{\exists\omega}$ is an indicator-function for numerical existence $\equiv$ for all $n$, if $n$ is a proof of $(\exists x \in \omega)A(x)$, then $f_{\exists\omega}(n)$ is a number $k$ for which $\vdash A(\boldsymbol{k})$.

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With this definition, I was asking in [**1**] whether **Z** possesses primitive recursive indicator-functions. I showed that no $f_{\exists\omega}$ was primitive recursive, but was emboldened by some unpublished work of Staples on 'combinator realizability' to conjecture that $f_\vee$ and $f_\exists$ could be chosen primitive recursive. The purpose of this note is to prove that for *no* **T** can we find $f_\vee$ or $f_{\exists\omega}$ which are provably recursive functions in **T** (let alone primitive recursive). The problem for $f_\exists$ remains open for the particular system **Z** of [**1**], but in general $f_\exists$ *can* be primitive recursive (e.g. if **T**=classical **ZF**+ $(V=L)$).

THEOREM. *Let $f_\vee$ be an indicator-function for* **T**. *Then $f_\vee$ is not provably recursive in* **T**.

PROOF. Suppose it were; i.e. suppose that, for some number $e$,

$$f_\vee(n) = U(\mu y)T(e, n, y)$$

and

$$\vdash(\forall x)(\exists y)T(\boldsymbol{e}, x, y).$$

Let $h_i(n)$ be a provably recursive function of **T** which enumerates all primitive recursive functions. Define (formally in **T**)

$$\Delta \equiv \{n \in \omega \mid f_\vee h_n(n) \neq 0\}.$$

Then

$$\vdash(\forall x \in \omega)(x \in \Delta \vee \neg x \in \Delta).$$

Let $h_k$ be a primitive recursive function such that, for each number $n$, $h_k(n)$ is a proof of $\boldsymbol{n} \in \Delta \vee \neg \boldsymbol{n} \in \Delta$.

Then

$$\begin{aligned} k \in \Delta &\rightarrow f_\vee h_k(k) = 1 \\ &\rightarrow h_k(k) \text{ proves } (\boldsymbol{k} \in \Delta \vee \neg \boldsymbol{k} \in \Delta) \wedge \vdash \neg \boldsymbol{k} \in \Delta \\ &\rightarrow k \notin \Delta \quad \text{(since } \mathbf{T} \text{ is consistent).} \end{aligned}$$

Conversely

$$\begin{aligned} k \notin \Delta &\rightarrow f_\vee h_k(k) = 0 \\ &\rightarrow h_k(k) \text{ proves } (\boldsymbol{k} \in \Delta \vee \neg \boldsymbol{k} \in \Delta) \wedge \vdash \boldsymbol{k} \in \Delta \\ &\rightarrow k \in \Delta \quad \text{(since } \mathbf{T} \text{ is consistent).} \end{aligned}$$

This is a contradiction. Q.E.D.

COROLLARY 1. **T** *cannot prove the disjunction-property for* **T**.

PROOF. If $\vdash(\forall\ulcorner A\urcorner\ulcorner B\urcorner)$ ($\ulcorner A\urcorner$, $\ulcorner B\urcorner$ closed and Thm $\ulcorner A\vee B\urcorner\rightarrow$Thm$\ulcorner A\urcorner\vee$ Thm$\ulcorner B\urcorner$) then

$$\begin{aligned} f_\vee \equiv \lambda x k(\mu y)[&(x \text{ is not a proof of any } \ulcorner A \vee B\urcorner \wedge y = 0) \\ &\vee (x \text{ is a proof of some } \ulcorner A \vee B\urcorner \\ &\qquad\qquad \wedge\, l(y) \text{ proves } \ulcorner A\urcorner \wedge k(y) = 0) \\ &\vee (x \text{ is a proof of some } \ulcorner A \vee B\urcorner \\ &\qquad\qquad \wedge\, l(y) \text{ proves } \ulcorner B\urcorner \wedge k(y) = 1)] \end{aligned}$$

would be an indicator-function provably recursive in **T**, contradicting the theorem. (Here $k$ and $l$ are the inverses of a primitive recursive pairing-function.)

COROLLARY 2. *$f_{\exists_\omega}$ is not provably recursive in* **T**.

PROOF. We have $f_\vee(n)=f_{\exists_\omega}b(n)$, where $b$ is a primitive recursive function such that if $m$ proves $A\vee B$, then $b(m)$ proves $(\exists x)((x=0\wedge A)\vee(x=1\wedge B))$. If $f_{\exists_\omega}$ were provably recursive, so would be $f_\vee$, contradicting the theorem.

COROLLARY 3. **T** *cannot prove the existence-property for numerals for* **T**.

Proof from Corollary 2 as Corollary 1 was proved from the theorem.

As we said above, the corresponding results for $f_\exists$ fail unless some additional conditions are placed on **T**. The principal open problem is to formulate these conditions and prove or disprove that they apply to systems like that of [**1**].

REFERENCE

**1.** J. Myhill, *Some properties of intuitionistic Zermelo-Frankel set-theory*, Proceedings of the Logic Conference at Cambridge, August 1971.

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