A NOTE ON INDICATOR-FUNCTIONS

J. MYHILL

Abstract. A system has the existence-property for abstracts (existence property for numbers, disjunction-property) if whenever $\vdash (\exists x)A(x)$, $\vdash A(t)$ for some abstract $(t)$ ($\vdash A(n)$ for some numeral $n$; if whenever $\vdash A \lor B$, $\vdash A$ or $\vdash B$. $(\exists x)A(x)$, $A$, $B$ are closed).

We show that the existence-property for numbers and the disjunction property are never provable in the system itself; more strongly, the (classically) recursive functions that encode these properties are not provably recursive functions of the system. It is however possible for a system (e.g., $\mathbf{ZF} + V = L$) to prove the existence-property for abstracts for itself.

In [1], I presented an intuitionistic form $\mathbf{Z}$ of Zermelo-Frankel set-theory (without choice and with weakened regularity) and proved for it the disjunction-property (if $\vdash A \lor B$ (closed), then $\vdash A$ or $\vdash B$), the existence-property (if $\vdash (\exists x)A(x)$ (closed), then $\vdash A(t)$ for a (closed) comprehension term $t$) and the existence-property for numerals (if $\vdash (\exists x \in \omega)A(x)$ (closed), then $\vdash A(n)$ for a numeral $n$). In the appendix to [1], I enquired if these results could be proved constructively; in particular if we could find primitive recursively from the proof of $A \lor B$ whether it was $A$ or $B$ that was provable, and likewise in the other two cases.

Discussion of this question is facilitated by introducing the notion of indicator-functions in the sense of the following

Definition. Let $T$ be a consistent theory which contains Heyting arithmetic (possibly by relativization of quantifiers). Then (where $f_\vee$, $f_3$, $f_\omega : \omega \rightarrow \omega$)

$f_\vee$ is an indicator-function for disjunction$\equiv$for all $n$, $f_\vee(n)$ is 0 or 1, and if $n$ is (the Gödel-number of) a proof of $A \lor B$, then $f_\vee(n) = 0$ implies $\vdash A$ while $f_\vee(n) = 1$ implies $\vdash B$;

$f_3$ is an indicator-function for existence$\equiv$for all $n$, if $n$ is a proof of $(\exists x)A(x)$, then $f_3(n)$ is the Gödel-number of a term $t$ for which $\vdash A(t)$; and

$f_\omega$ is an indicator-function for numerical existence$\equiv$for all $n$, if $n$ is a proof of $(\exists x \in \omega)A(x)$, then $f_\omega(n)$ is a number $k$ for which $\vdash A(k)$.
With this definition, I was asking in [1] whether $Z$ possesses primitive recursive indicator-functions. I showed that no $f_{3\omega}$ was primitive recursive, but was emboldened by some unpublished work of Staples on 'combinator realizability' to conjecture that $f_\nu$ and $f_3$ could be chosen primitive recursive. The purpose of this note is to prove that for no $T$ can we find $f_\nu$ or $f_{3\omega}$ which are provably recursive functions in $T$ (let alone primitive recursive). The problem for $f_3$ remains open for the particular system $Z$ of [1], but in general $f_3$ can be primitive recursive (e.g. if $T =$ classical $ZF + (\forall = L)$).

**Theorem.** Let $f_\nu$ be an indicator-function for $T$. Then $f_\nu$ is not provably recursive in $T$.

**Proof.** Suppose it were; i.e. suppose that, for some number $e$,

$$f_\nu(n) = U(\mu y)T(e, n, y)$$

and

$$\vdash (\forall x)(\exists y)T(e, x, y).$$

Let $h_\nu(n)$ be a provably recursive function of $T$ which enumerates all primitive recursive functions. Define (formally in $T$

$$\Delta \equiv \{n \in \omega \mid f_\nu h_\nu(n) \neq 0\}.$$  

Then

$$\vdash (\forall x \in \omega)(x \in \Delta \forall \forall x \in \Delta).$$

Let $h_k$ be a primitive recursive function such that, for each number $n$, $h_k(n)$ is a proof of $n \in \Delta \forall \forall n \in \Delta$.

Then

$$k \in \Delta \to f_\nu h_k(k) = 1$$

$$\to h_k(k) \text{ proves } (k \in \Delta \forall \forall k \in \Delta) \land \vdash \forall k \in \Delta$$

$$\to k \notin \Delta \quad (\text{since } T \text{ is consistent}).$$

Conversely

$$k \notin \Delta \to f_\nu h_k(k) = 0$$

$$\to h_k(k) \text{ proves } (k \in \Delta \forall \forall k \in \Delta) \land \vdash k \in \Delta$$

$$\to k \in \Delta \quad (\text{since } T \text{ is consistent}).$$

This is a contradiction. Q.E.D.

**Corollary 1.** $T$ cannot prove the disjunction-property for $T$. 
Proof. If $\vdash (\forall A \forall B \neg) (\neg A \land \neg B)$ closed and Thm $\neg A \lor B \rightarrow$ Thm $\neg A \lor \neg B$ then

$$f_v \equiv \lambda x k(\mu y)[(x \text{ is not a proof of any } \neg A \lor B \land y = 0)$$

$$\lor (x \text{ is a proof of some } \neg A \lor B)$$

$$\land l(y) \text{ proves } \neg A \land k(y) = 0)$$

$$\lor (x \text{ is a proof of some } \neg A \lor B)$$

$$\land l(y) \text{ proves } \neg B \land k(y) = 1)]$$

would be an indicator-function provably recursive in $T$, contradicting the theorem. (Here $k$ and $l$ are the inverses of a primitive recursive pairing-function.)

Corollary 2. $f_{3^\omega}$ is not provably recursive in $T$.

Proof. We have $f_v(n) = f_{3^\omega} b(n)$, where $b$ is a primitive recursive function such that if $m$ proves $A \lor B$, then $b(m)$ proves $(\exists x)((x = 0 \land A) \lor (x = 1 \land B))$. If $f_{3^\omega}$ were provably recursive, so would be $f_v$, contradicting the theorem.

Corollary 3. $T$ cannot prove the existence-property for numerals for $T$.

Proof from Corollary 2 as Corollary 1 was proved from the theorem.

As we said above, the corresponding results for $f_3$ fail unless some additional conditions are placed on $T$. The principal open problem is to formulate these conditions and prove or disprove that they apply to systems like that of [1].

Reference