A CLASS OF PARTIALLY ORDERED LINEAR ALGEBRAS
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Abstract. We consider a special type of partially ordered linear algebra which is like an algebra of real-valued functions. We show that various natural properties characterize this type of algebra. These natural properties relate the algebraic and order structures to each other.

A pola (denoted by $A$) is a real linear associative algebra which is partially ordered so that it is a directed partially ordered linear space and $0 \leq xy$ whenever $x, y \in A$, $0 \leq x$, $0 \leq y$. We also assume that $A$ has a multiplicative identity $1^0$. A Dedekind $\sigma$-complete pola (dsc-pola) $A$ is one having the property: if $x_n \in A$, $0 \leq \cdots \leq x_2 \leq x_1$, then $\text{inf}\{x_n\}$ exists. Order convergence is defined as usual. A dsc-pola $A$ has the Archimedean property: if $x, y \in A$ and $nx \leq y$ for every positive integer $n$, then $x \leq 0$. For more details and examples see the references.

The simple example of interest to us here is the dsc-pola $A$ of all real-valued functions defined on some nonempty set, where the algebraic operations and the partial order are defined pointwise. We note that $A$ has the following property:

$P_1$: If $x \in A$ and $x \geq 1$, then $x$ has an inverse and $x^{-1} \geq 0$.

If we now consider an arbitrary dsc-pola $A$ which has property $P_1$, then we can show that $A$ is much like an algebra of real-valued functions; however, the operations may only be defined "almost everywhere" (see example 5 of [4]). Some of the basic properties are given in the following theorem.

Theorem 1. If $A$ is a dsc-pola which has property $P_1$, then multiplication of elements in $A$ is commutative and $A$ is a lattice. Furthermore, $x^2 \geq 0$ for all $x \in A$ and if $y \in A$ and $y \geq 0$, then there exists a unique $z \in A$ such that $z^2 = y$ and $z \geq 0$.

This theorem was proved by the author but it appears as the necessary introduction to the thesis of his former student, T. Dai, who showed in

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addition that $A$ (having property $P_1$) is an $f$-ring [1, p. 403]. The reader is referred to Dai's paper [2] for examples and proofs. The purpose of this paper is to show that various natural properties for a dsc-pola $A$ imply that $A$ has property $P_1$.

**Lemma 1.** Let $A$ be a dsc-pola. If $x \leq 1$ and there exists $y \geq 0$ such that $1 \leq xy$ or $1 \leq yx$, then $x$ has an inverse and $x^{-1} \geq 1$. From this it follows that if $1 \leq u \leq v$ and $v$ has an inverse and $v^{-1} \geq 0$, then $u$ has an inverse and $u^{-1} \geq 0$.

For the proof see Proposition 3 and its corollary in [3].

**Lemma 2.** Let $A$ be a dsc-pola which has the property: if $x \in A$ and $x \geq 0$, then there exists a sequence $\{x_n\}$ of elements from $A$ such that $0 \leq x_n \leq n1$ for all $n$ and o-lim $x_n = x$. Then $A$ has property $P_1$.

For the proof see (v) of Lemma 1.6.6 in [2].

The following two properties concern one-sided factoring of one element by another.

**Theorem 2.** Let $A$ be a dsc-pola which has the property: if $y_1, y_2 \in A$ and $0 \leq y_1 \leq y_2$, then there exists $w \in A$ such that $w \geq 0$ and $y_1 = wy_2$. Then $A$ has property $P_1$.

**Proof.** Take any $x \in A$ such that $x \geq 1 \geq 0$. Hence, there exists $w \in A$ such that $w \geq 0$ and $wx = 1$. Since $w \geq 0$ and $x \geq 1$, we have $w \leq 1$. Using Lemma 1, we see that $x^{-1} = w \geq 0$.

**Theorem 3.** Let $A$ be a dsc-pola which has the property: if $y_1, y_2 \in A$ and $1 \leq y_1 \leq y_2$, then there exists $w \in A$ such that $w \geq 1$ and $wy_1 = y_2$. Then $A$ has property $P_1$.

**Proof.** Take any $x \in A$ such that $x \geq 1$. Since $1 \leq x \leq x + 1$, there exists $w \in A$ such that $w \geq 1$ and $wx = x + 1$. Hence, $(w - 1)x = 1$, where $w - 1 \geq 0$. Thus, $w - 1 \leq 1$ and we may again use Lemma 1 to show that $x^{-1} = w - 1 \geq 0$.

The following is a decomposition property for multiplication in a dsc-pola which is commutative.

**Theorem 4.** Let $A$ be a commutative dsc-pola which has the property: if $y \in A$, $y \geq 0$ and $0 \leq w \leq y^2$, then there exist elements $u, v \in A$ such that $0 \leq u \leq y$, $0 \leq v \leq y$ and $uv = w$. Then $A$ has property $P_1$.

**Proof.** Take any $x, y \in A$ such that $1 \leq x \leq x + 1 \leq y$. Thus, $0 \leq y^2 - 1 \leq y^2$. Hence, we may find $u, v \in A$ such that $0 \leq u \leq y$, $0 \leq v \leq y$, and $uv = y^2 - 1$. We see easily that $1 = y(y - u) + u(y - v)$. We remark that this is the only place we use commutativity. Using the inequalities given above, one can easily show that $0 \leq y - u \leq 1$, $u \geq 1$ and then $0 \leq y - v \leq 1$. Since
\[ 1 + (y-u)(y-v) = y(2y-u-v) \] and \[ 0 \leq (y-u)(y-v) \leq 1, \] we see that \((\frac{1}{2})(y(2y-u-v)) \leq 1\) and then by using Lemma 1, we obtain \([y(2y-u-v)]^{-1} \geq 0\). By using Lemma 1 twice, one can show first that \(y^{-1} \geq 0\) and then \(x^{-1} \geq 0\).

At the end of the paper a counterexample will be given to show that \(A\) must be commutative in the previous theorem. However, we can drop commutativity if we use a stronger decomposition property as follows.

**Theorem 5.** Let \(A\) be a dsc-pola which has the property: if \(y \in A, y \geq 0\) and \(0 \leq w \leq y^2\), then there exists \(u \in A\) such that \(0 \leq u \leq y\) and \(u^2 = w\). Then \(A\) has property \(P_1\).

**Proof.** Take any \(x, y \in A\) such that \(1 \leq x \leq x+1 \leq y\). Thus, \(0 \leq y^2 - 1 \leq y^2\). Hence, we may find \(u \in A\) such that \(0 \leq u \leq y\) and \(u^2 = y^2 - 1\). We see easily that \(1 = y(y-u) + (y-u)u\). Using the inequalities given above, one can easily show that \(0 \leq y - u \leq 1\). Since \(0 \leq y - u\), we see that \(y(y-u) \leq 1\) and \(y(y-u)(1+u) = y(y-u) + y(y-u)u \geq 1\). Using Lemma 1 twice, we can first show that \([y(y-u)]^{-1} \geq 0\) and then \(y^{-1} \geq 0\). Using Lemma 1 again, we can show that \(x^{-1} \geq 0\).

Next we consider an order-reversing property for left inverses.

**Theorem 6.** Let \(A\) be a dsc-pola which has the property: if \(y_1, y_2 \in A\) and \(1 \leq y_1 \leq y_2\), then there exist \(w_1, w_2 \in A\) such that \(w_2 \leq w_1\) and \(w_1 y_1 = w_2 y_2 = 1\). Then \(A\) has property \(P_1\).

**Proof.** Take any \(x \in A\) such that \(1 \leq x \leq x+1\). There exist \(w_1, w_2 \in A\) such that \(w_2 \leq w_1\) and \(w_1 x = w_2 (x+1) = 1\). Hence, \((w_1 - w_2) x = w_2\) and \(w_1 - w_2 \mid x \mid (x+1) = 1\). Since \(x(x+1) \geq 1\) and \(0 \leq w_1 - w_2\), we have \(w_1 - w_2 \leq 1\). Using Lemma 1 twice, we can first show that \([x(x+1)]^{-1} = w_1 - w_2 \geq 0\) and then \(x^{-1} \geq 0\).

The next property concerns generalized inverses.

**Theorem 7.** Let \(A\) be a dsc-pola which has the property: if \(z \in A\) and \(z \geq 1\), then there exists \(w \in A\) such that \(w \geq 0\) and \(zwz = z\). Then \(A\) has property \(P_1\).

**Proof.** From the above it follows that if \(v \in A, v \geq 1\) and \(v\) has an inverse, then \(v^{-1} \geq 0\). Let us now take \(w\) and \(z\) as in the statement of the theorem. If we put \(u = wz\), then \(0 \leq u = u^2\). Since \(1 + nu \geq 1\) and \(1 + nu\) has an inverse for every positive integer \(n\), we obtain \(0 \leq (1 + nu)^{-1} = 1 - (n/n+1)u\) for all \(n\). Using the Archimedean property, we obtain \(wz = u \leq 1\). Since \(hwz = z \geq 1\), we obtain \((wz)^{-1} \geq 1\) by using Lemma 1. By again using Lemma 1 we obtain \(z^{-1} \geq 0\).

The following question is unanswered.
**Question.** Let $A$ be a dsc-pola which has the property: if $x \in A$, then $x^2 \geq 0$. Does $A$ have property $P_1$?

We can answer this question in certain special cases.

**Theorem 8.** Let $A$ be a dsc-pola which has the properties: $A$ is a lattice and if $x \in A$, then $x^2 \geq 0$. Then $A$ has property $P_1$.

**Proof.** Take any $z \in A$ such that $z \geq 0$. Since $(2n1 - z)^2 \geq 0$, we get $0 \leq z \leq n1 + (1/4)n z^2$ for every positive integer $n$. Since $A$ is a lattice, we may write $z = z_n + w_n$, where $0 \leq z_n \leq n1$ and $0 \leq w_n \leq (1/4)n z^2$ for all $n$. Thus, $0 \leq z_n = z$. Using Lemma 2, we see that $A$ has property $P_1$.

Some definitions are necessary for the next two theorems. An element $u \in A$ is called an order unit if $u \geq 0$ and if for any $x \in A$ there exists a real number $\alpha$ such that $-\alpha u \leq x \leq \alpha u$. A dsc-pola $A$ is said to have the Perron-Frobenius (PF) property if for every $x \in A$, $x \geq 0$, there exists a real number $\lambda > 0$ such that $\lambda 1 - x$ has an inverse and $(\lambda 1 - x)^{-1} \geq 0$. The name of this property is justified in [4].

**Theorem 9.** Let $A$ be a dsc-pola. If $A$ is finite-dimensional, then $A$ has an order unit. If $A$ has an order unit, then $A$ has the PF property. If $A$ has the PF property, then $A$ has the large inverse property.

**Proof.** The first implication is a consequence of two facts: $A$ is finite dimensional and $A$ is directed. The proof of the second implication can be found in Theorem 6 of [4]. The proof of the third implication can be found in Proposition 3 of [3].

**Theorem 10.** Let $A$ be a dsc-pola which has the large inverse property and also has the property: if $z \in A$, then $z^2 \geq 0$. Then $A$ has property $P_1$.

**Proof.** Take any $x \in A$ such that $x \geq 1$. Since $A$ has the large inverse property, there exists $y \in A$ such that $x \leq y$ and $y$ has an inverse. From the second property we obtain $0 \leq (y^{-1})^2 = (y^2)^{-1}$. Since $1 \leq x \leq y \leq y^2$, we can use Lemma 1 to show that $x^{-1} \geq 0$.

The final two theorems concern special assumptions about the way an element can be expressed as the difference of two nonnegative elements.

**Theorem 11.** Let $A$ be a dsc-pola which has the property: if $x \in A$, then there exists $a \in A$ such that $0 \leq a \leq 1$, $ax \geq 0$ and $(1-a)x \leq 0$. Then $A$ has property $P_1$. 


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Proof. Take any \( z \in A \) such that \( z \geq 0 \). Next select \( a_n \in A \) such that \( 0 \leq a_n \leq 1 \), \( a_n(n1−z) \geq 0 \) and \( (1−a_n)(n1−z) \leq 0 \) for every positive integer \( n \). Hence, \( 0 \leq a_n z \leq na_n \leq n1 \) and \( 0 \leq n(1−a_n) \leq (1−a_n)z \). From the latter inequalities and the fact that \( 0 \leq 1−a_n \leq 1 \), we obtain \( 0 \leq n(1−a_n) \leq z \) and then \( 0 \leq z−a_n z \leq (1/n)z^2 \). Putting \( z_n = a_n z \), we see that \( 0 \leq z_n \leq n1 \) and \( \text{o-lim} z_n = z \). We may now use Lemma 2 to show that \( A \) has property \( P_1 \).

The last theorem was proved by the author but credit is due Ralph Gellar. He proved a slightly weaker theorem which inspired the author to work on the following theorem.

Theorem 12. Let \( A \) be a dsc-pola which has the property: if \( x \in A \), then there exist \( y, z \in A \) such that \( y \geq 0, z \geq 0, yz = 0 \) and \( x = y−z \). Then \( A \) has property \( P_1 \). (Gellar also assumed that \( zy = 0 \).)

Proof. The key idea involved is that if \( a \in A \) and \( 0 \leq a^2 \leq a \), then \( a \leq 1 \).

We first prove this fact.

There exist \( b, c \in A \) such that \( b \geq 0, c \geq 0, bc = 0 \) and \( 1−a = b−c \). Since \( a^2 \leq a \), we have \( 0 \leq a−a^2 = a(1−a) = ab−c \), which means that \( 0 \leq ac \leq ab \). Hence, \( 0 \leq ac^2 \leq abc = 0 \), which means that \( ac^2 = 0 \). Since \( 1 \leq 1+c = a+b \), we obtain \( 0 \leq c^2 \leq ac^2 = 0 \), which means that \( c^2 = 0 \). Now there exist \( d, e \in A \) such that \( d \geq 0, e \geq 0, de = 0 \) and \( 1−c = d−e \). Therefore, \( 1 \leq 1+e = c+d \) so that \( e \leq ce \). Hence, \( 0 \leq e \leq ce \leq c^2 e = 0 \), which means that \( e = 0 \). It follows that \( 0 \leq 1−c \) so that \( 0 \leq (1−c)^n = 1−nc \) for every positive integer \( n \). From the Archimedean property it follows that \( c = 0 \), which means that \( a \leq 1 \).

Now take any \( h \in A \) such that \( h \geq 0 \). For each positive integer \( n \) there exist \( y_n, z_n \in A \) such that \( y_n \geq 0, z_n \geq 0, y_n z_n = 0 \) and \( n1−h = y_n − z_n \). Hence, \( 0 \leq y_n \leq y_n(y_n + h) = y_n(n1 + z_n) = ny_n \) for all \( n \). Thus, \( 0 \leq (1/n)^2 y_n^2 \leq (1/n)y_n \) so that \( 0 \leq y_n \leq n1 \) for all \( n \). The last inequality is obtained from the result of the preceding paragraph. Since \( y_n \leq n1 \), we obtain \( z_n \leq h \) for all \( n \). Now \( nz_n \leq (n1 + z_n)z_n = (y_n + h)z_n = hz_n \leq h^2 \) for all \( n \), which means that if we define \( h_n = n1 − y_n \), then \( 0 \leq h−h_n \leq (1/n)h^2 \). Thus, \( 0 \leq h_n \leq n1 \) and \( \text{o-lim} h_n = h \). Using Lemma 2, we see that \( A \) has property \( P_1 \).

Counterexamples. Let \( M \) be the real linear algebra of all 2-by-2 matrices in upper triangular form, where all entries are real. If \( M \) is partially ordered entry by entry, then \( M \) is a dsc-pola which is not commutative. The reader is invited to use \( M \) to verify that the order conditions are necessary in Theorems 2, 3, 5, 6 and 7. For example, look at the proof of Theorem 2. If we take any \( x \in M \) such that \( x \geq 1 \), then there exists \( w \in M \) such that \( wx = 1 \), but it may happen that \( w \) not \( \geq 0 \). The dsc-pola \( M \) has the property described in Theorem 4 but it is not commutative. Note that \( M \) is a lattice and has the large inverse property but it does not have
the other property needed in Theorems 8 and 10. Also $M$ does not have the properties described in Theorems 11 and 12.

References


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