ON HARDY'S INEQUALITY AND LAPLACE TRANSFORMS IN WEIGHTED REARRANGEMENT INVARIANT SPACES
KENNETH F. ANDERSEN

Abstract. Hardy's well-known inequality relating the norm of a function and the norm of its integral mean in the Lebesgue spaces $L^p(\mu)$, $d\mu(t)=t^{\sigma-1}dt$, is extended to the class of rearrangement invariant spaces $X(\mu)$. These spaces include, for example, the $L^p(\mu)$, the Lorentz and the Orlicz spaces. As an application, necessary and sufficient conditions are obtained for an operator related to the Laplace transform to be bounded as a linear operator between rearrangement invariant spaces of $\mu$-measurable functions.

For $\sigma>0$, we write $d\mu(t)=t^{\sigma-1}dt$ and denote by $L^p(\mu)$ the space of $\mu$-measurable functions on $(0, \infty)$ for which the norm

$$\|f\|_{p, \mu} = \left\{ \left( \int_0^\infty |f(t)|^p d\mu(t) \right)^{1/p}, \quad 1 \leq p < \infty, \right.$$  

$$\text{ess sup } |f(t)|, \quad p = \infty,$$

is finite, and if $X$ and $Y$ are Banach spaces, $[X, Y]$ will denote the space of bounded linear operators from $X$ into $Y$. We abbreviate $[X, X]=[X]$. Let the operators $P$ and $P'$ be defined by

$$(Pf)(s) = \int_0^s f(t) \, dt \quad \text{and} \quad (P'f)(s) = \int_s^\infty f(t) \frac{dt}{t}$$

whenever the required integrals exist for all $s>0$. Then Hardy's celebrated theorem [4, pp. 245-246] may be stated in the form:

**Theorem** Suppose $1 \leq p < \infty$. Then $P \in [L^p(\mu)]$ if $p>\sigma$ and $P' \in [L^p(\mu)]$ if $\sigma>0$.

In this paper we determine necessary and sufficient conditions which allow the $L^p(\mu)$ spaces which appear in Hardy's theorem to be replaced by a function space of a more general class. The class of spaces with which we deal possess the property of rearrangement invariance and include, for
example, the $L^p(\mu)$ spaces, the Lorentz and the Orlicz spaces. We apply our results to the Laplace transform considered as a linear operator between rearrangement invariant spaces.

Following the approach of [2] we assume that $X(m)$, $m$ denoting Lebesgue measure on $(0, \infty)$, is a Banach space of Lebesgue measurable functions on $(0, \infty)$ whose norm $\| \cdot \|_{X(m)}$ is rearrangement invariant in the sense that two functions which are equimeasurable with respect to $m$ have the same norm. The rearrangement invariant space $X(\mu)$ then consists of those $\mu$-measurable functions on $(0, \infty)$ for which $f^* \in X(m)$ and the norm in $X(\mu)$ is given by $\|f\|_{X(\mu)} = \|f^*\|_{X(m)}$. Here, as usual, $f^*$ denotes the nonnegative, nonincreasing rearrangement of $f$ which is equimeasurable with $f$ in the sense that

$$\mu\{t : |f(t)| > y\} = m\{t : f^*(t) > y\} \quad (y > 0).$$

For more details see [1], [2].

The upper index $\alpha$ and the lower index $\beta$ corresponding to the rearrangement invariant space $X$ was defined by Boyd [2] in terms of the function $h(s, X, Y)$, where for rearrangement invariant spaces $X$ and $Y$, $h(s, X, Y)$ denotes the norm in $[X(m), Y(m)]$ of the dilation operator $E_s : (E_s f)(t) = f(st)$. Note that if $X \subseteq Y$, then $h(1, X, Y)$ is finite. This follows from the closed graph theorem and [1, Definition 1.1(iv)].

We have the following:

**Theorem 1.** Let $X$ be a rearrangement invariant space with upper index $\alpha$ and lower index $\beta$. Then

(i) $P \in [X(\mu)]$ if and only if $\alpha \sigma < 1$.

(ii) $P' \in [X(\mu)]$ if and only if $\beta > 0$.

**Theorem 2.** Let $X$ and $Y$ be rearrangement invariant spaces and let $h(s) = h(s, X, Y)$. Then

(i) The condition $X \subseteq Y$ is necessary for $P \in [X(\mu), Y(\mu)]$ and the condition $\int_0^1 h(s^\alpha) \, ds < \infty$ is sufficient for $P \in [X(\mu), Y(\mu)]$.

(ii) The condition $X \subseteq Y$ is necessary for $P' \in [X(\mu), Y(\mu)]$ and the condition $\int_1^\infty h(s^\alpha) \, ds < \infty$ is sufficient for $P' \in [X(\mu), Y(\mu)]$.

For the Lorentz space $L^{p,q}$ the indices are given by $\alpha = \beta = 1/p$, so in particular, taking $p=q$ in Theorem 1 we recover Hardy's theorem. The indices for the various Lorentz spaces $\Lambda(\varphi, p)$, $M(\varphi, p)$ and the Orlicz spaces $L_{M\Phi}$, $L_{\Phi}$ have been computed by Boyd [1]. We leave to the reader the application of those results to our Theorem 1.

For the particular case $\sigma = 1$, Theorems 1 and 2 were obtained by Boyd [1] and applied to a study of the Hilbert transform. Here, we give one application of our results to a transform which is related to the Laplace
transform \( \mathcal{L} \). Let the transform \( T \) be given by

\[
(Tf)(s) = \frac{1}{s} (\mathcal{L}f) \left( \frac{1}{s} \right) = \int_0^\infty e^{-t/s} f(t) \frac{dt}{s} \quad (s > 0).
\]

It is well known that \( T \in [L^p(\mu)] \) if and only if \( p > \sigma \), indeed, the case \( \sigma = 1 \) is suggested in [5, p. 397, Ex. 16] as an application of Hardy's theorem. We prove the following:

**Theorem 3.** Let \( X \) and \( Y \) be rearrangement invariant spaces. Then \( T \in [X(\mu), Y(\mu)] \) if and only if \( \rho > \alpha \), indeed, the case \( \alpha = 1 \) is suggested in [5, p. 397, Ex. 16] as an application of Hardy's theorem.

We prove the following:

**Theorem 3.** Let \( X \) and \( Y \) be rearrangement invariant spaces. Then \( T \in [X(\mu), Y(\mu)] \) if and only if \( \rho > \alpha \), indeed, the case \( \alpha = 1 \) is suggested in [5, p. 397, Ex. 16] as an application of Hardy's theorem.

**Corollary 1.** If \( X \) has upper index \( \alpha \), then \( T \in [X(\mu)] \) if and only if \( \alpha \sigma < 1 \).

**Corollary 2.** \( X \subseteq Y \) is a necessary condition and \( \int_0^1 h(s^\alpha, X, Y) \, ds < \infty \) is sufficient for \( T \in [X(\mu), Y(\mu)] \).

**Corollary 3.** If \( \sigma < 1 \), then \( T \in [X(\mu), Y(\mu)] \) if and only if \( X \subseteq Y \).

Corollaries 1 and 2 follow immediately from the theorems, and according to [1, p. 605, Lemma 3.2] \( sh(s, X, Y) \leq h(1, X, Y) \) for \( 0 < s < 1 \), so if \( \sigma < 1 \) and \( X \subseteq Y \) we have

\[
\int_0^1 h(s^\sigma, X, Y) \, ds \leq h(1, X, Y) \int_0^1 s^{-\sigma} \, ds < \infty
\]

and Corollary 3 then follows from Corollary 2.

The theorems depend on the following lemma which is adapted from §3 of [1] and which deals with operators of the following form: Let \( a(t) \) be nonnegative and measurable on \( (0, \infty) \). Define

\[
(Kf)(s) = \int_0^\infty a(t) f(st) \, dt
\]

and

\[
(\tilde{K}f)(s) = \int_0^\infty \frac{1}{t^{1/\sigma}} a(t^{1/\sigma}) f(st) \frac{dt}{t}
\]

whenever the required integrals exist for all \( s > 0 \).

**Lemma.** Suppose \( X \) and \( Y \) are rearrangement invariant spaces and \( K, \tilde{K} \) are as defined above.

(a) \( K \in [X(\mu), Y(\mu)] \) if and only if \( \tilde{K} \in [X(m), Y(m)] \), indeed, \( K \) and \( \tilde{K} \) have the same norm in the respective spaces.

(b) If \( c = \int_0^\infty a(s) h(s^\alpha, X, Y) \, ds < \infty \), then \( K \in [X(\mu), Y(\mu)] \) with \( \|K\| \leq c \).

(c) If \( K \in [X(\mu), Y(\mu)] \) and \( A(s) = \int_0^s a(t) \, dt \) then \( A(s)h(s^\alpha, X, Y) \leq \|K\| \).

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
If \( a(t) > 0 \) on a set of positive measure and \( K \in [X(\mu), Y(\mu)] \) then \( X \subseteq Y \).

**Proof.** Let \( f \in X(\mu) \) and define, for \( t > 0 \),

\[
\phi(t) = (\tau f)(t) = f(t^\sigma / \sigma).
\]

Then, for each \( y > 0 \),

\[
m \{ t : \phi(t) > y \} = \mu \{ t : |g(t)| > y \} = m \{ t : |f(t^\sigma / \sigma)| > y \} = m \{ t : |f(t)| > y \}
\]

so that for any rearrangement invariant space \( Z \) we have

\[
\| \phi \|_{Z(\mu)} = \| \phi \|_{Z(\mu)} = \| \phi \|_{Z(m)} = \| f \|_{Z(m)}.
\]

Now,

\[
(K \phi)((a(\sigma)^{1/\sigma}) = \int_0^\infty a(t) g((a(\sigma)^{1/\sigma}) dt = \int_0^\infty a(t) f(s^\sigma) dt = \int_0^\infty \frac{1}{\sigma} t^{1/\sigma} a(t) f(st) \frac{dt}{t} = (\tilde{K} f)(s)
\]

that is, \( K(\phi) = \phi(\tilde{K} f) \) and hence, from (1),

\[
\| \tilde{K} f \|_{Y(\mu)} = \| \phi(\tilde{K} f) \|_{Y(\mu)} = \| K(\phi) \|_{Y(\mu)}
\]

from which (a) follows. Now according to [1, Theorem 3.1], \( \tilde{K} \in [X(m), Y(m)] \) whenever

\[
\int_0^\infty a(s) h(s, X, Y) ds < \infty,
\]

so (b) follows from (a). Finally, (c) and (d) follow easily from (a) and [1, Lemma 3.3].

Note that the operators \( P, P', \) and \( T \) are, for appropriate choices of \( a(t) \), of the form \( K \) in the lemma. In particular, Theorem 2 follows immediately from the lemma, and we now prove Theorems 1 and 3.

**Proof of Theorem 1.** According to (a) of the lemma, \( P \in [X(\mu)] \) if and only if \( \tilde{P} \in [X(m)] \) where

\[
(\tilde{P} f)(s) = \int_0^1 \frac{1}{\sigma} t^{1/\sigma} f(st) \frac{dt}{t},
\]

so (i) follows from (50) of [2]. Again by the lemma, \( P' \in [X(\mu)] \) if and only if \( P = \sigma \tilde{P}' \in [X(m)] \). Now if \( X' \) is the associate space of \( X \) with upper index \( \alpha' \), then \( P' \in [X(m)] \) if and only if \( P \in [X'(m)] \), and since \( \beta = 1 - \alpha' \), (ii) follows from (i).

**Proof of Theorem 3.** Since

\[
(P \mid f)(s) = \int_0^1 |f(st)| dt \leq \epsilon \int_0^\infty e^{-\epsilon t} |f(st)| dt = e(T \mid f)(s)
\]
we get $P \in [X(\mu), Y(\mu)]$ whenever $T \in [X(\mu), Y(\mu)]$. On the other hand,

$$T|f| \leq P|f| + Q|f|$$

where

$$(Qf)(s) = \int_1^\infty e^{-s}f(st) \, dt \quad (s > 0),$$

so we need only show that $Q \in [X(\mu), Y(\mu)]$ whenever $P \in [X(\mu), Y(\mu)]$. Now if $P \in [X(\mu), Y(\mu)]$ then $X \subseteq Y$ by Theorem 2 so that

$$\int_1^\infty e^{-s}h(s^*, X, Y) \, ds \leq h(1, X, Y) \int_1^\infty e^{-s} \, ds < \infty$$

and $Q \in [X(\mu), Y(\mu)]$ by (b) of the lemma.

REFERENCES


