INDECOMPOSABLE CONTINUA IN STONE-ČECH COMPACTIFICATIONS

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Abstract. We show that if $Y$ is a continuum irreducible from $a$ to $b$, which is connected im Kleinen and first countable at $b$, and if $X = Y - \{b\}$, then $\beta X - X$ is an indecomposable continuum. Examples are given showing that both first countability and connectedness im Kleinen are needed here. We also show that $\beta(0, 1) - \{0, 1\}$ has a strong near-homogeneity property.

1. Introduction. In [2] and [3] it is shown that if $X = [0, 1)$ then $\beta X - X$ is an indecomposable continuum; here $\beta X$ is the Stone-Čech compactification of $X$. In [7], Dickman showed that among locally connected spaces, $[0, 1)$ is essentially the only such space. In this paper we exhibit other types of spaces $X$ with this property. We shall also show that for $X = [0, 1)$, $\beta X - X$ is stably almost homogeneous, a concept to be defined below.

The set function $T$ has been studied and applied in [1], [5], [6], [8], [9], [11], and [14]. We follow these papers in writing $T(p)$ for $T(\{p\})$. This set function will be used in the argument at one point and familiarity with it is assumed. Familiarity with [10], [12], [13], and [15] is also assumed. If we write $X = A \cup B$ sep, then we mean that $\text{Cl}(A) \cap B = \emptyset$ and $A \cap \text{Cl}(B) = \emptyset$ while neither $A$ nor $B$ is empty. By $f : X \rightarrow Y$, we mean $f$ is a homeomorphism of $X$ onto $Y$.

2. Indecomposable continua in $\beta X$.

Lemma 1. There is a covariant functor $\beta$ on the category of Tychonoff spaces and continuous maps such that for any Tychonoff space $X$, $\beta X$ is the Stone-Čech compactification of $X$ and if $f : X \rightarrow Y$ then $\beta f : \beta X \rightarrow \beta Y$ is the unique extension of $f$ induced by $f$ treated as a map from $X$ to $\beta Y$.

Notation. If $X$ is a Tychonoff space, let $\gamma X = \beta X - X$. If $f$ is a continuous map from $X$ to $Y$, let $\gamma f$ denote $\beta f | \gamma X$.

Definition. Let $Y$ be a compact Hausdorff continuum irreducible from $a$ to $b$ such that $Y$ is both connected im Kleinen and first countable

Received by the editors November, 1971 and, in revised form, September 8, 1972.


Key words and phrases. Remainders of compactifications.

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at \( b \). Let \( X = Y - \{ b \} \). Then we call the topological pair \(( Y, X)\) a wave from \( a \) to \( b \).

By stringing together indecomposable continua, a wave \(( Y, X)\) can be constructed such that \( Y \) is not connected im Kleinen at any point of \( X \).

**Lemma 2.** If \( Y \) is a compact Hausdorff continuum irreducible from \( a \) to \( b \) and \( x \in Y \), \( T(x) \) either separates \( a \) from \( b \), contains \( a \), or contains \( b \). In case \( T(x) \) separates \( a \) from \( b \), \( Y - T(x) \) has exactly two components, \( A \) and \( B \), where \( a \in A \) and \( b \in B \), and both \( T(x) \cup A \) and \( T(x) \cup B \) are proper subcontinua of \( Y \) containing \( a \) and \( b \) respectively.

**Remark on proof.** This lemma can be established using standard techniques and Theorem 1.10 of [14], since each \( x \in Y \) different from \( a \) and \( b \) weakly separates \( a \) from \( b \).

**Lemma 3.** If \( Y \) is a compact Hausdorff continuum irreducible from \( a \) to \( b \) and \( W \subseteq Y \) is a continuum with \( b \in \text{Int } W \), then \( W - \{ b \} \) is connected.

**Proof.** If \( W - \{ b \} = M_0 \cup N_0 \), let \( M = M_0 \cup \{ b \} \); \( N = N_0 \cup \{ b \} \). Then \( b \) lies in the boundary of \( M \) and \( N \) and, by Theorem 6 of [15, p. 194], each of \( M \) and \( N \) is nowhere dense, so that \( M \cup N = W \) is nowhere dense also, a contradiction.

**Lemma 4.** If \(( Y, X)\) is a wave from \( a \) to \( b \), and \( Z \) is a Hausdorff compactification of \( X \), then \( Z - X \) is a Hausdorff continuum.

**Proof.** Since \( Y \) is connected im Kleinen and first countable at \( b \), there exists a denumerable collection of continua \( \{ N_i \}_{i=1}^{\infty} \) such that for each \( i \), \( b \in \text{Int } (N_i) \) and \( N_{i+1} \subseteq N_i \) and \( \bigcap_{i=1}^{\infty} N_i = \{ b \} \). It is readily seen that

\[
Z - X = \text{Cl}_Z (N_i - \{ b \}) - N_i = \bigcap_{i=1}^{\infty} \text{Cl}_Z (N_i - \{ b \}).
\]

Then, since each \( N_i - \{ b \} \) is connected by Lemma 3, \( Z - X \) is an intersection of a monotone collection of continua.

**Lemma 5.** Let \( X \) be a compact Hausdorff space, \( b \in X \), \( \{ b \} \) a component of \( X \), and suppose \( X \) is first countable at \( b \) and \( \{ b_i \}_{i=1}^{\infty} \) is a sequence in \( X - \{ b \} \) converging to \( b \). Then there exist two closed subsets \( A \) and \( B \) of \( X \) such that \( A \cup B = X \), \( A \cap B = \{ b \} \), and each of \( A \) and \( B \) contains infinitely many (that is, a subsequence) of the \( b_i \)'s.

**Proof.** It is readily seen that there exists a neighborhood basis \( \{ N_i \}_{i=1}^{\infty} \) at \( b \) consisting of closed and open sets such that \( N_{j+1} \subseteq N_j \) for each \( j \) and \( N_1 = X \); by passing to a subset if necessary, we may suppose that each
$N_j - N_{j+1}$ contains at least one of the $b_i$'s. Then set

$$A = \{b\} \cup \bigcup_{j=1}^{\infty} (N_{2j-1} - N_{2j}), \quad B = \{b\} \cup \bigcup_{j=1}^{\infty} (N_{2j} - N_{2j+1}).$$

Then $A$ and $B$ have the desired properties.

**Lemma 6.** If $(Y, X)$ is a wave from $a$ to $b$ and $W$ is a nondegenerate subcontinuum of $Y$ containing $b$, then $b \in \text{Int } W$.

**Proof.** Suppose not. Then let $p \in W$, $p \neq b$. Since $b \notin T(p)$, by connectedness im Kleinen at $b$, it follows that either $a \in T(p)$ or $Y - T(p) = A \cup B$ sep, where $a \in A$ and $b \in B$. If $a \in T(p)$, $T(p) \cup W$ is a proper subcontinuum of $Y$ containing both $a$ and $b$; if $a \notin T(p)$, $A \cup T(p) \cup W$ is such a continuum, and in either case we have a contradiction.

**Corollary 1.** If $(Y, X)$ is a wave from $a$ to $b$ and $M$ is a closed subset of $Y$ with $b \in M$ but $b \notin \text{Int } M$, $\{b\}$ is a component of $M$.

**Lemma 7.** If $Y$ is a compact Hausdorff space first countable at a point $b$, then $Y - \{b\}$ is normal.

**Proof.** Let $\{O_k\}_{k=1}^{\infty}$ be a countable basis of open neighborhoods at $b$. Then $Y - \{b\} = \bigcup_{k=1}^{\infty} (Y - O_k)$, so that $Y - \{b\}$ is sigma compact and hence Lindelöf. Then $Y - \{b\}$ is paracompact [10, p. 174, 6.5] and hence normal [10, p. 163, 2.2].

**Theorem 1.** If $(Y, X)$ is a wave from $a$ to $b$, then $yX$ is an indecomposable continuum.

**Proof.** By Lemma 4, $yX$ is a continuum. Suppose $F$ is a proper subcontinuum of $yX$ which contains an interior point $q$ with respect to $yX$. Let $p \in y(X) - F$. Let $U$ and $V$ be open sets in $yX$ with $\text{Cl}(U) \cap \text{Cl}(V) = \text{Cl}(U) \cap (yX - \text{Int } F) = \text{Cl}(V) \cap F = \emptyset$ while $p \in V$ and $q \in U$. This is possible by regularity.

Then $X \cap V$ and $X \cap U$ are open subsets of $X$ and hence of $Y$ since $X$ is open in $Y$. Let $\langle b_i \rangle_{i=1}^{\infty}$ be a sequence of points in $U \cap X$ converging in $Y$ to $b$. This is possible since $b \in \text{Cl}_Y(U \cap X)$ and $Y$ is first countable at $b$.

Then $\{b\}$ is a component of $Y - (V \cap X)$, by Corollary 1, and $\langle b_i \rangle$ is a sequence in $(Y - V) - \{b\}$ converging to $b$. By Lemma 6 there are two closed sets $A_0$ and $B_0$ such that $A_0 \cup B_0 = Y - V$, $A_0 \cap B_0 = \{b\}$, and each of $A_0$ and $B_0$ contains a subsequence of the $b_i$'s. Let $A = A_0 \cap X$, $B = B_0 \cap X$. Then $A$ and $B$ are disjoint closed subsets of $X$, and since $X$ is normal, disjoint closed sets lie in disjoint zero sets, and by Theorem 6.5 III of [12], $\text{Cl}_X(A) \cap \text{Cl}_X(B) = \emptyset$. Now since each of $A$ and $B$ contains infinitely many of the $b_i$'s, it follows that each of $\text{Cl}_X(A)$ and $\text{Cl}_X(B)$ contains
points of $\text{Cl}_{\beta X}(U) \cap \gamma X$, and hence points of $F$. Thus, since if $x \in \gamma X - \text{Cl}_{\beta X}(A \cup B)$, it follows that $x \in \text{Cl}_{\beta X}(V)$ and hence $x \notin F$, we have $F = (F \cap \text{Cl}_{\beta X}(A)) \cup (F \cap \text{Cl}_{\beta X}(B)) \text{ sep}$, so that $F$ is no continuum.

**Corollary 2 ([2] and [3]).** Let $X = [0, 1)$. Then $\gamma X$ is an indecomposable continuum.

**Example 1.** Let $L$ denote the long line, consisting of $\omega_1 \times [0, 1)$ with the lexicographic order, where $\omega_1$ is the first uncountable ordinal; we take the order topology on $L$. Then consider $L \times [0, 1]$ with the product topology. Let

$$X = \{((x, t), s) \in L \times [0, 1]: t = 0 \text{ or } t = s\}.$$

Let $Y = X \cup \{b\}$ be the one-point compactification of $X$. Then $Y$ is irreducible from $((0, 0), 1)$ to $b$ and is connected im Kleinen at $b$. $(Y, X)$ fails to be a wave from $a$ to $b$ because $Y$ is not first countable at $b$.

Standard techniques applied to continuous functions from $\omega_1$ to $[0, 1]$ yield the result that $\gamma X \cong [0, 1]$ in this case. Thus, first countability cannot be dispensed with in the hypothesis of Theorem 1. Connectedness im Kleinen also cannot be removed from the hypothesis of Theorem 1; the usual topologist's sin $1/x$ curve, with $b$ taken from the limit arc, yields a decomposable continuum as $\gamma X$.

**Lemma 8.** If $X$ is a Tychonoff space and $Z$ is any compactification of $X$ with inclusion map $i: X \rightarrow Z$, then $\gamma i(\gamma X) = Z - i(X)$.

**Remark on proof.** This is a special case of Theorem 6.12 of [12, p. 92].

**Lemma 9.** If $X$ and $Y$ are Tychonoff spaces and $f: X \rightarrow Y$, then $\gamma f : \gamma X \rightarrow \gamma Y$.

**Proof.** By Lemma 8, $\gamma f(\gamma X) = \gamma Y$ and since $\beta$ is a functor it follows that $\beta f$ is a homeomorphism since it has inverse $\beta(f^{-1})$. Then $\gamma f$ is a homeomorphism since it is a restriction of one.

**Lemma 10.** Let $X$ be a normal Hausdorff space and $A$ a closed subset of $X$ such that $X - A$ contains a closed but not compact subset of $X$. Then $\gamma X - \text{Cl}_{\beta X}(A)$ is a nonempty, open subset of $\gamma X$.

**Lemma 11.** Suppose $X$ is a Tychonoff space and $f: X \cong X$ is the identity inside some closed subset $V$ of $X$. Then $\gamma f: \gamma X \cong \gamma X$ is the identity inside $\gamma X \cap \text{Cl}_{\beta X}(V)$.

**Definition.** We say a topological space $X$ is **almost homogeneous** if for any $p, q \in X$, and any neighborhood $U$ of $q$ there is a homeomorphism $h: X \cong X$ such that $h(p) \in U$. If, in addition, we may choose $h$ to be the
identity on some nonempty open subset of $X$, we say $X$ is stably almost homogeneous.

**Theorem 2.** Let $X = [0, 1)$; then $\gamma X$ is a stably almost homogeneous indecomposable continuum.

**Proof.** Throughout this proof, $\text{Cl}$ denotes $\text{Cl}_\beta X$. Let $x, y \in \gamma X$ and let $V_0$ be any open set in $\gamma X$ containing $y$. Then $V_0 = V_1 \cap \gamma X$ for some $V_1$ open in $\beta X$. Then there exists a $V_2$ open in $\beta X$ such that $y \in V_2 \subseteq \text{Cl} V_2 \subseteq V_1$ and $x \notin \text{Cl} V_2$ unless $x = y$, in which case there is nothing to prove. Let $U_0$ be open in $\beta X$ with $x \in U_0$ and $\text{Cl} U_0 \cap \text{Cl} V_2 = \emptyset$. Now let $U = U_0 \cap X$ and $V = V_2 \cap X$. We shall assume, with no loss of generality, that $0 < \inf U < \inf V$.

Now, define four sequences $(p_n)_{n=1}^{\infty}$, $(q_n)_{n=1}^{\infty}$, $(r_n)_{n=1}^{\infty}$, and $(s_n)_{n=1}^{\infty}$ as follows: $p_1 = \inf U$. Whenever $p_i$ has been defined, set $q_i = \sup \{t \in U : [p_i, t] \cap V = \emptyset \}$. When $q_i$ has been defined, set $r_i = \inf \{t \in V : t > q_i \}$. When $r_i$ has been defined, set $s_i = \sup \{t \in V : r_i \cap U = \emptyset \}$. When $s_i$ has been defined, set $p_{i+1} = \inf \{t \in U : t > s_i \}$. This completes the recursive definition of the four sequences. They have the following properties: $p_i < q_i < r_i < s_i < p_{i+1}$ for each $i$; the limit in $[0, 1]$ of each sequence is $1$, $U \subseteq \bigcup_{i=1}^{\infty} [p_i, q_i]$, and $V \subseteq \bigcup_{i=1}^{\infty} [r_i, s_i]$. We now choose two more sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ so that, for each $i$, $r_i < x_i < y_i < s_i$ and the closed interval $[x_i, y_i]$ is a subset of $V$. Finally we choose two more sequences $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ such that $a_1 = 0$; $0 < b_1 < p_1$, and for $i > 1$ we choose $s_i < a_{i+1} < b_{i+1} < p_{i+1}$. Now define $h : X \rightarrow X$ as follows: For each $i$,

1. $h$ is the identity on $[a_i, b_i]$,
2. $h$ maps the interval $[b_i, p_i]$ linearly onto $[b_i, x_i]$,
3. $h$ maps $[p_i, q_i]$ linearly onto $[x_i, y_i]$,
4. $h$ maps $[q_i, a_{i+1}]$ linearly onto $[y_i, a_{i+1}]$.

Then $h(U) \subseteq V$, and hence $\beta h(\text{Cl}(U)) \subseteq \text{Cl}(V)$, and since $x \in \text{Cl}(U)$, $\beta h(x) \in \text{Cl}(V) \subseteq \text{Cl}(V_2) \subseteq V_1$, and $\beta h(x) = \gamma h(x) \in V_0$ as desired. Furthermore, $\gamma h$ is the identity inside the set $\gamma X \cap \text{Cl}(\bigcup_{i=1}^{\infty} [a_i, b_i])$, which contains a nonvoid open subset of $\gamma X$ by Lemma 10, setting the closed set $\bigcup_{i=1}^{\infty} [b_i, a_{i+1}]$ equal to $A$ in the lemma.

**References**


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