INDECOMPOSABLE CONTINUA IN STONE-ČECH
COMPACTIFICATIONS

DAVID P. BELLAMY AND LEONARD R. RUBIN

Abstract. We show that if \( Y \) is a continuum irreducible from \( a \) to \( b \), which is connected im Kleinen and first countable at \( b \), and if \( X = Y - \{ b \} \), then \( \beta X - X \) is an indecomposable continuum. Examples are given showing that both first countability and connectedness im Kleinen are needed here. We also show that \( \beta(0, 1) - \{0, 1\} \) has a strong near-homogeneity property.

1. Introduction. In [2] and [3] it is shown that if \( X = [0, 1) \) then \( \beta X - X \) is an indecomposable continuum; here \( \beta X \) is the Stone-Cech compactification of \( X \). In [7], Dickman showed that among locally connected spaces, \([0, 1)\) is essentially the only such space. In this paper we exhibit other types of spaces \( X \) with this property. We shall also show that for \( X = [0, 1) \), \( \beta X - X \) is stably almost homogeneous, a concept to be defined below.

The set function \( T \) has been studied and applied in [1], [5], [6], [8], [9], [11], and [14]. We follow these papers in writing \( T(p) \) for \( T(\{p\}) \). This set function will be used in the argument at one point and familiarity with it is assumed. Familiarity with [10], [12], [13], and [15] is also assumed. If we write \( X = A \cup B \) sep, then we mean that \( Cl(A) \cap B = \emptyset \) and \( A \cap Cl(B) = \emptyset \) while neither \( A \) nor \( B \) is empty. By \( f : X \cong Y \), we mean \( f \) is a homeomorphism of \( X \) onto \( Y \).

2. Indecomposable continua in \( \beta X \).

Lemma 1. There is a covariant functor \( \beta \) on the category of Tychonoff spaces and continuous maps such that for any Tychonoff space \( X \), \( \beta X \) is the Stone-Čech compactification of \( X \) and if \( f : X \to Y \) then \( \beta f : \beta X \to \beta Y \) is the unique extension of \( f \) induced by \( f \) treated as a map from \( X \) to \( \beta Y \).

Notation. If \( X \) is a Tychonoff space, let \( \gamma X = \beta X - X \). If \( f \) is a continuous map from \( X \) to \( Y \), let \( \gamma f \) denote \( \beta f | \gamma X \).

Definition. Let \( Y \) be a compact Hausdorff continuum irreducible from \( a \) to \( b \) such that \( Y \) is both connected im Kleinen and first countable.

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at \( b \). Let \( X = Y - \{ b \} \). Then we call the topological pair \(( Y, X)\) a wave from \( a \) to \( b \).

By stringing together indecomposable continua, a wave \(( Y, X)\) can be constructed such that \( Y \) is not connected im Kleinen at any point of \( X \).

**Lemma 2.** If \( Y \) is a compact Hausdorff continuum irreducible from \( a \) to \( b \) and \( x \in Y \), \( T(x) \) either separates \( a \) from \( b \), contains \( a \), or contains \( b \). In case \( T(x) \) separates \( a \) from \( b \), \( Y - T(x) \) has exactly two components, \( A \) and \( B \), where \( a \in A \) and \( b \in B \), and both \( T(x) \cup A \) and \( T(x) \cup B \) are proper sub-continua of \( Y \) containing \( a \) and \( b \) respectively.

**Remark on proof.** This lemma can be established using standard techniques and Theorem 1.10 of [14], since each \( x \in Y \) different from \( a \) and \( b \) weakly separates \( a \) from \( b \).

**Lemma 3.** If \( Y \) is a compact Hausdorff continuum irreducible from \( a \) to \( b \) and \( W \subseteq Y \) is a continuum with \( b \in \text{Int} \, W \), then \( W - \{ b \} \) is connected.

**Proof.** If \( W - \{ b \} = M \cup N \) sep, let \( M = M_0 \cup \{ b \} ; N = N_0 \cup \{ b \} \). Then \( b \) lies in the boundary of \( M \) and \( N \) and, by Theorem 6 of [15, p. 194], each of \( M \) and \( N \) is nowhere dense, so that \( M \cup N = W \) is nowhere dense also, a contradiction.

**Lemma 4.** If \(( Y, X)\) is a wave from \( a \) to \( b \), and \( Z \) is a Hausdorff compactification of \( X \), then \( Z - X \) is a Hausdorff continuum.

**Proof.** Since \( Y \) is connected im Kleinen and first countable at \( b \), there exists a denumerable collection of continua \( \{ N_i \}_{i=1}^\infty \) such that for each \( i \), \( b \in \text{Int}(N_i) \) and \( N_{i+1} \subseteq N_i \) and \( \bigcap_{i=1}^\infty N_i = \{ b \} \). It is readily seen that

\[
Z - X = \text{Cl}_Z(N_i - \{ b \}) - N_i = \bigcap_{i=1}^\infty \text{Cl}_Z(N_i - \{ b \}).
\]

Then, since each \( N_i - \{ b \} \) is connected by Lemma 3, \( Z - X \) is an intersection of a monotone collection of continua.

**Lemma 5.** Let \( X \) be a compact Hausdorff space, \( b \in X \), \( \{ b \} \) a component of \( X \), and suppose \( X \) is first countable at \( b \) and \( (b_i)_{i=1}^\infty \) is a sequence in \( X - \{ b \} \) converging to \( b \). Then there exist two closed subsets \( A \) and \( B \) of \( X \) such that \( A \cup B = X \), \( A \cap B = \{ b \} \), and each of \( A \) and \( B \) contains infinitely many (that is, a subsequence) of the \( b_i \)’s.

**Proof.** It is readily seen that there exists a neighborhood basis \( \{ N_j \}_{j=1}^\infty \) at \( b \) consisting of closed and open sets such that \( N_{j+1} \subseteq N_j \) for each \( j \) and \( N_1 = X \); by passing to a subset if necessary, we may suppose that each
Then \( A \) and \( B \) have the desired properties.

**Lemma 6.** If \((Y, X)\) is a wave from \(a\) to \(b\) and \(W\) is a nondegenerate subcontinuum of \(Y\) containing \(b\), then \(b \in \text{Int } W\).

**Proof.** Suppose not. Then let \(p \in W\), \(p \neq b\). Since \(b \notin T(p)\), by connectedness im Kleinen at \(b\), it follows that either \(a \in T(p)\) or \(Y - T(p) = A \cup B\) sep, where \(a \in A\) and \(b \in B\). If \(a \in T(p)\), \(T(p) \cup W\) is a proper subcontinuum of \(Y\) containing both \(a\) and \(b\); if \(a \notin T(p)\), \(A \cup T(p) \cup W\) is such a continuum, and in either case we have a contradiction.

**Corollary 1.** If \((Y, X)\) is a wave from \(a\) to \(b\) and \(M\) is a closed subset of \(Y\) with \(b \in M\) but \(b \notin \text{Int } M\), \(\{b\}\) is a component of \(M\).

**Lemma 7.** If \(Y\) is a compact Hausdorff space first countable at a point \(b\), then \(Y - \{b\}\) is normal.

**Proof.** Let \(\{O_k\}_{k=1}^\infty\) be a countable basis of open neighborhoods at \(b\). Then \(Y - \{b\} = \bigcup_{k=1}^\infty (Y - O_k)\), so that \(Y - \{b\}\) is sigma compact and hence Lindelöf. Then \(Y - \{b\}\) is paracompact [10, p. 174, 6.5] and hence normal [10, p. 163, 2.2].

**Theorem 1.** If \((Y, X)\) is a wave from \(a\) to \(b\), then \(\gamma X\) is an indecomposable continuum.

**Proof.** By Lemma 4, \(\gamma X\) is a continuum. Suppose \(F\) is a proper subcontinuum of \(\gamma X\) which contains an interior point \(q\) with respect to \(\gamma X\). Let \(p \in \gamma(X) - F\). Let \(U\) and \(V\) be open sets in \(\beta X\) with \(\overline{\text{Cl}(U)} \cap \overline{\text{Cl}(V)} = \overline{\text{Cl}(U)} \cap (\gamma(X) - \text{Int } F) = \overline{\text{Cl}(V)} \cap F = \emptyset\) while \(p \in V\) and \(q \in U\). This is possible by regularity.

Then \(X \cap V\) and \(X \cap U\) are open subsets of \(X\) and hence of \(Y\) since \(X\) is open in \(Y\). Let \(\langle b_i \rangle_{i=1}^\infty\) be a sequence of points in \(U \cap X\) converging in \(Y\) to \(b\). This is possible since \(b \in \overline{\text{Cl}_Y(U \cap X)}\) and \(Y\) is first countable at \(b\).

Then \(\{b\}\) is a component of \(Y - (V \cap X)\), by Corollary 1, and \(\langle b_i \rangle\) is a sequence in \((Y - V) - \{b\}\) converging to \(b\). By Lemma 6 there are two closed sets \(A_0\) and \(B_0\) such that \(A_0 \cup B_0 = Y - V\), \(A_0 \cap B_0 = \{b\}\), and each of \(A_0\) and \(B_0\) contains a subsequence of the \(b_i\)'s. Let \(A = A_0 \cap X\), \(B = B_0 \cap X\). Then \(A\) and \(B\) are disjoint closed subsets of \(X\), and since \(X\) is normal, disjoint closed sets lie in disjoint zero sets, and by Theorem 6.5 III of [12], \(\overline{\text{Cl}_X(A)} \cap \overline{\text{Cl}_X(B)} = \emptyset\). Now since each of \(A\) and \(B\) contains infinitely many of the \(b_i\)'s, it follows that each of \(\overline{\text{Cl}_X(A)}\) and \(\overline{\text{Cl}_X(B)}\) contains
points of Cl_{pX}(U) ∩ γX, and hence points of F. Thus, since if x ∈ γX−
Cl_{pX}(A ∪ B), it follows that x ∈ Cl_{pX}(V) and hence x ∉ F, we have F =
(F ∩ Cl_{pX}(A)) ∪ (F ∩ Cl_{pX}(B)) sep, so that F is no continuum.

**Corollary 2 ([2] and [3]).** Let X = [0, 1). Then γX is an indecom-
posable continuum.

**Example 1.** Let L denote the long line, consisting of ω₁ × [0, 1) with
the lexicographic order, where ω₁ is the first uncountable ordinal; we take
the order topology on L. Then consider L × [0, 1] with the product
topology. Let

\[ X = \{(x, t), s) \in L \times [0, 1] : t = 0 \text{ or } t = s \}. \]

Let Y = X ∪ {b} be the one-point compactification of X. Then Y is
irreducible from ((0, 0), 1) to b and is connected im Kleinen at b. (Y, X)
fails to be a wave from a to b because Y is not first countable at b.

Standard techniques applied to continuous functions from ω₁ to [0, 1]
yield the result that γX ≃ [0, 1] in this case. Thus, first countability cannot
be dispensed with in the hypothesis of Theorem 1. Connectedness im Kleinen also cannot be removed from the hypothesis of Theorem 1; the
usual topologist's sin 1/x curve, with b taken from the limit arc, yields a
decomposable continuum as γX.

**Lemma 8.** If X is a Tychonoff space and Z is any compactification of X
with inclusion map i : X → Z, then γY(γX) = Z − i(X).

**Remark on proof.** This is a special case of Theorem 6.12 of [12,
p. 92].

**Lemma 9.** If X and Y are Tychonoff spaces and f : X → Y, then γf :
γX ≃ γ Y.

**Proof.** By Lemma 8, γf(γX) = γ Y and since β is a functor it follows
that βf is a homeomorphism since it has inverse β(f⁻¹). Then γf is a
homeomorphism since it is a restriction of one.

**Lemma 10.** Let X be a normal Hausdorff space and A a closed subset of
X such that X − A contains a closed but not compact subset of X. Then
γX − Cl_{pX}(A) is a nonempty, open subset of γX.

**Lemma 11.** Suppose X is a Tychonoff space and f : X ≃ X is the identity
inside some closed subset V of X. Then γf : γX ≃ γX is the identity inside
γX ∩ Cl_{pX}(V).

**Definition.** We say a topological space X is almost homogeneous if
for any p, q ∈ X, and any neighborhood U of q there is a homeomorphism
h : X ≃ X such that h(p) ∈ U. If, in addition, we may choose h to be the
identity on some nonempty open subset of $X$, we say $X$ is stably almost homogeneous.

**Theorem 2.** Let $X=\{0, 1\}$; then $\gamma X$ is a stably almost homogeneous indecomposable continuum.

**Proof.** Throughout this proof, $\text{Cl}$ denotes $\text{Cl}_{\beta X}$. Let $x, y \in \gamma X$ and let $V_0$ be any open set in $\gamma X$ containing $y$. Then $V_0 = V_1 \cap \gamma X$ for some $V_1$ open in $\beta X$. Then there exists a $V_2$ open in $\beta X$ such that $y \in V_2 \subseteq \text{Cl} V_2 \subseteq V_1$ and $x \notin \text{Cl} V_2$ unless $x=y$, in which case there is nothing to prove. Let $U_0$ be open in $\beta X$ with $x \in U_0$ and $\text{Cl} U_0 \cap \text{Cl} V_2 = \emptyset$. Now let $U = U_0 \cap X$ and $V = V_2 \cap X$. We shall assume, with no loss of generality, that $0 < \inf U < \inf V$.

Now, define four sequences $(p_n)_{n=1}^{\infty}$, $(q_n)_{n=1}^{\infty}$, $(r_n)_{n=1}^{\infty}$, and $(s_n)_{n=1}^{\infty}$ as follows: $p_1 = \inf U$. Whenever $p_i$ has been defined, set $q_i = \sup \{t \in U : [p_i, t] \cap V = \emptyset\}$. When $q_i$ has been defined, set $r_i = \inf \{t \in V : t > q_i\}$. When $r_i$ has been defined, set $s_i = \sup \{t \in V : [r_i, t] \cap U = \emptyset\}$. When $s_i$ has been defined, set $p_{i+1} = \inf \{t \in U : t > s_i\}$. This completes the recursive definition of the four sequences. They have the following properties: $p_1 < q_1, r_1 < s_1 < p_{i+1}$ for each $i$; the limit in $[0, 1]$ of each sequence is 1, $U \subseteq \bigcup_{i=1}^{\infty} [p_i, q_i]$, and $V \subseteq \bigcup_{i=1}^{\infty} [r_i, s_i]$. We now choose two more sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ so that, for each $i$, $r_i < x_i < y_i < s_i$ and the closed interval $[x_i, y_i]$ is a subset of $V$. Finally we choose two more sequences $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ such that $a_1 = 0$; $0 < b_1 < p_1$, and for $i > 1$ we choose $s_i < a_{i+1} < b_{i+1} < p_{i+1}$. Now define $h : X \to X$ as follows: For each $i$,

1. $h$ is the identity on $[a_i, b_i]$,
2. $h$ maps the interval $[b_i, p_i]$ linearly onto $[b_i, x_i]$,
3. $h$ maps $[p_i, q_i]$ linearly onto $[x_i, y_i]$,
4. $h$ maps $[q_i, a_{i+1}]$ linearly onto $[y_i, a_{i+1}]$.

Then $h(U) \subseteq V$, and hence $\beta h(\text{Cl}(U)) \subseteq \text{Cl}(V)$, and since $x \in \text{Cl}(U)$, $\beta h(x) \in \text{Cl}(V) \subseteq \text{Cl}(V_2) \subseteq V_1$, and $\beta h(x) = y h(x) \in V_0$ as desired. Furthermore, $\gamma h$ is the identity inside the set $\gamma X \cap \text{Cl}(\bigcup_{i=1}^{\infty} [a_i, b_i])$, which contains a nonvoid open subset of $\gamma X$ by Lemma 10, setting the closed set $\bigcup_{i=1}^{\infty} [b_i, a_{i+1}]$ equal to $A$ in the lemma.

**References**


